Irreducibility, Resource Relatedness and Survival in Equilibrium with Individual Non-convexities

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Abstract

Standard results in general equilibrium theory, such as existence and the second efficiency and core equivalence theorems, are most easily proved for compensated equilibria. A new condition establishes that, even with individual non-convexities, in compensated equilibrium any agent with a cheaper feasible net trade is also in uncompensated equilibrium. Some generalizations of McKenzie's irreducibility assumption are then presented. They imply that (almost) no agent is at a cheapest point, so the easier and more general results for compensated equilibria become true for uncompensated equilibria. Survival of all consumers in uncompensated equilibrium also depends on satisfying an additional assumption that is similar to irreducibility.

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IRREDUCIBLE ECONOMIES

1. Introduction

1.1. General Equilibrium Theory in the 1950's

The heyday of mathematical economics in general, and of general equilibrium theory in particular, was surely the decade of the 1950's. For the first time general logically consistent models of a competitive market economy were developed and existence of a competitive equilibrium was proved. First Arrow (1951) proved the two efficiency theorems of welfare economics. The first of these theorems shows that any competitive equilibrium is Pareto efficient. The second shows that, under assumptions such as convexity and continuity which have since become standard, almost any Pareto efficient allocation can be achieved through competitive markets in general equilibrium, provided that lump-sum redistribution is used to ensure that each individual has the appropriate amount of puchasing power.

After these efficiency theorems, the next step was to give a proper proof of the existence of equilibrium for a given exogenous distribution of purchasing power. McKenzie (1954a) gave such a proof for a world economy with continuous aggregate demand functions for each nation. This was done using newly available fixed point theorems in mathematics such as Kakutani's (1941).[1] McKenzie's proof used arguments which were obviously capable of significant generalization. Also, since he was using Graham's (1948) model of international trade in which whole nations were being considered as economic units, McKenzie (1954a, b) naturally saw no need to assume that each nation has the preferences of a representative national consumer. Indeed, Gorman's (1953) work the year before had already shown how restrictive such an assumption would be, even if it were possible to classify all individuals as nationals of their own nations in which they live permanently.

In the very next issue of *Econometrica*, Arrow and Debreu (1954) gave the first published general proof of existence for an economy in which preferences and endowments, rather than demand functions, are exogenous. Of course, one could well argue that this model is somewhat more appropriate for general existence proofs than Graham's, because

[1] Despite considering demand correspondences rather than (single-valued) demand functions, in his later general existence proof McKenzie (1959) was able to use just Brouwer's fixed point theorem for continuous functions, rather than Kakutani's theorem for correspondences. See Khan (1989) for further details.

one should perhaps consider a model of international trade in which each consumer in each nation is treated as a separate economic unit. Anyway, these results were then extended, in rapid succession, in such works as those of McKenzie (1955, 1959, 1961), Gale (1955), Nikaido (1956, 1957), and Debreu (1959, 1962).

Although no excuse for revisiting the 1950's is required on this occasion, I shall nevertheless provide one. It should come as no surprise that the arguments of these path-breaking papers can be somewhat streamlined thirty or more years later. Nor that the assumptions can be considerably relaxed. In the process of doing so, however, it has become very useful to adopt an approach first hinted at in Arrow and Debreu (1954), and then put into practice in Arrow and Hahn (1971).[2] This requires first demonstrating either that a particular Pareto efficient or core allocation is a compensated equilibrium, or that such an equilibrium exists. Only at a second stage is it shown that the compensated equilibrium must be an ordinary or uncompensated equilibrium. An important part of this approach to extending the results of general equilibrium theory involves concepts closely related to McKenzie's (1959, 1961, 1981, 1987) irreducibility assumption. Yet the key role played by this assumption (or by Arrow and Hahn's similar notion of "resource relatedness") has often been neglected.[3] Nor has much attention been given to the different possible variations of this assumption, or even (apart from Spivak, 1978) to whether significant weakenings may still be possible.

1.2. From Compensated to Uncompensated Equilibrium

Indeed, following the approach of Arrow (1951), it is almost trivial to prove the first efficiency theorem. In its weak form, this states that, without any assumptions at all, any competitive equilibrium allocation is weakly Pareto efficient, in the sense that there is no other physically feasible allocation which moves all agents simultaneously to a preferred allocation. In its stronger and more familiar form, the first efficiency theorem states that if agents have locally non-satiated preferences, then any competitive equilibrium allocation is Pareto efficient in the sense that there is no other physically feasible allocation which moves some agent or agents to a preferred allocation without simultaneously moving others

[2] See also Moore (1975).

[3] For example, it is not discussed at all in Duffie and Sonnenschein's (1989) survey, or in Hildenbrand and Kirman's (1988) advanced textbook.

to a dispreferred allocation. Equally trivial is the theorem which says that any competitive equilibrium allocation is in the core.

The departures from standard assumptions which interest us include non-convexity of feasible sets or of preferences, the possible non-survival of some individuals, boundary problems, and exceptional cases such as that discussed by Arrow (1951). Such departures from standard assumptions can do nothing to perturb these very simple and robust results. They do, however, create serious difficulties for the existence of the competitive equilibrium which is predicated in the first efficiency theorem. They may also make the core empty, or lead to violations of the second efficiency and core equivalence theorems. One of the principal aims of our work, therefore, is to generalize the conditions under which these important results in general equilibrium theory retain their validity, especially in continuum economies for which individual non-convexities may become smoothed out.

To this end, an approach which appears to be very helpful is to break the problem up into two parts. It turns out to be relatively easy to prove weak versions of the existence, second efficiency, and core equivalence theorems, using a weaker notion of *compensated* competitive equilibrium. This terminology is apparently due to Arrow and Hahn (1971), though the concept can be found in Arrow and Debreu (1954, Section 5.3.3). In compensated equilibrium every agent is minimizing expenditure at equilibrium prices, subject to not falling below an upper contour set associated with the preference relation (cf. McKenzie, 1957). Whereas uncompensated equilibrium is the more familiar and natural kind of equilibrium in which every agent is maximizing preferences subject to the budget constraint that net expenditure at equilibrium prices cannot be positive. This distinction is brought out in Section 2. The weak version of the existence theorem is simply that a compensated equilibrium exists. The weak version of the second efficiency theorem is that (almost) any Pareto efficient allocation can be achieved as a compensated equilibrium, provided that appropriate lump-sum redistribution of wealth has occurred. The weak version of the core equivalence theorem is that any allocation in the core of a continuum economy must be a compensated equilibrium. These three weak theorems have been stated and proved in various ways by numerous different authors. The assumptions under which they are valid do not necessarily require individuals' feasible sets to be convex, nor do they require that all individuals survive in compensated equilibrium.

Of course, these weak theorems concerning compensated equilibrium still leave us with the problem of proving the existence of an uncompensated competitive equilibrium, as well as the usual versions of the second efficiency and core equivalence theorems which refer to uncompensated equilibria. One way to prove these results, obviously, is to pass as directly as possible from a compensated equilibrium to an uncompensated equilibrium. This is precisely where irreducibility becomes important. Section 3 discusses some of the precursors to the idea of irreducibility, especially the extremely implausible interiority assumption used originally by Arrow (1951) and by Arrow and Debreu (1954). As is well known, the main function of this assumption is to ensure directly that all agents have cheaper points in their feasible sets. If individuals have convex feasible sets and continuous preferences, this ensures that any compensated equilibrium is also an uncompensated equilibrium. But since this work is intended to allow non-convex feasible sets, a different assumption concerning particular cones generated by the feasible set and by the set of preferred net trade vectors is used to establish this standard conclusion. This is also taken up in Section 3, along with an aggregate interiority condition which rules out Arrow's (1951) exceptional case by ensuring that at least one individual has a cheaper feasible net trade vector.

Thereafter, Section 4 presents a "convexified non-oligarchy" condition which proves adequate to show that a particular compensated equilibrium allocation is an uncompensated equilibrium. For the special case of "linear exchange" economies, in which each consumer's indifference map consists of parallel line segments in the non-negative orthant, this condition is virtually equivalent to irreducibility, according to Gale's (1957) and Eaves' (1976) definition, or to Gale's (1976) definition of "self-sufficiency." The results of this section confirm that an important part of a more appealing sufficient condition for compensated equilibria to become uncompensated equilibria involves considering the extent to which different agents in the economy are able to benefit from one another's resources. Moreover, a result due to Spivak (1978) is extended to show that this convexified non-oligarchy condition is necessary as well as sufficient for every agent to have a cheaper point in any compensated equilibrium with lump-sum redistribution.

The convexified non-oligarchy condition ensures that for a specific compensated equilibrium allocation and corresponding equilibrium price vector, every consumer has a cheaper feasible point and so, given the additional assumptions we shall make in Section 4, is actually in uncompensated equilibrium. To ensure existence of an uncompensated equilibrium, it would therefore be sufficient to have every allocation which is weakly Pareto superior to autarky be convexified non-oligarchic. This sufficient condition is unnecessarily strong, however. For existence theorems it is enough to consider those compensated equilibria without lump-sum transfers in which the equilibrium allocation and price vector imply that each consumer's budget hyperplane passes through the autarky allocation. In order to ensure that at these equilibria each consumer has a cheaper point, it is enough to ensure that each consumer's endowment is not a cheapest point. To this end, McKenzie's concept of "irreducibility" considers the benefits of replicating the consumers in the economy so that their initial endowments can then be exploited by the existing agents. This is the topic of Section 5. It points out how McKenzie's fundamental contribution was incompletely anticipated in the latter part of Arrow and Debreu (1954). As mentioned above, an earlier paper by Gale (1957) contained a somewhat related idea for linear exchange economies. Irreducibility was duly recognized by Debreu (1962). Not surprisingly, Arrow and Hahn's (1971) later concept of resource relatedness also turns out to be closely related. Section 5 presents a generalized version of irreducibility which encompasses virtually all previous conditions of this kind,[4] and is also suitable for economies in which individuals may have non-convex feasible sets. The new condition, moreover, is the weakest possible which guarantees that, in any compensated equilibrium without lump-sum transfers, each consumer has a cheaper point.

In addition, irreducibility was extended by Moore (1975) and McKenzie (1981, 1987) to finite economies in which consumers may have to rely on others to survive. The last part of Section 5 generalizes their sufficient conditions which ensure that any compensated equilibrium in the economy is actually an uncompensated equilibrium in which all consumers are able to survive.

Finally, following Hildenbrand's (1972) modification of irreducibility, Section 6 shows how to extend the analysis to continuum economies. There is no need to consider convexified versions of the earlier non-oligarchy and interdependence conditions because the continuum of agents itself ensures convexity.

[4] The sole exception of which I am aware is Gale's (1976) special condition for linear exchange economies.

Proofs of many of the main results which are very similar to each other are gathered together in Section 7. Section 8 contains a few concluding remarks.

2. Preliminaries

2.1. Basic Assumptions

2.1.1. A Finite Set of Agents

Assume that there is a finite set A of economic agents, with typical member denoted by a.

2.1.2. The Commodity Space

Assume that there is a fixed finite set G of physical commodities, so that the commodity space is the finite dimensional Euclidean space $\Re^G$.

2.1.3. Consumers' Feasible Sets

Next, assume that every agent $a \in A$ has a fixed feasible set $X_a \subset \Re^G$ of net trade vectors x_a satisfying $0 \in X_a$. If agent $a \in A$ happens to have a fixed endowment $\omega_a \in \Re^G$, then each $x_a \in X_a$ is equal to the difference $c_a - \omega_a$ between a feasible consumption vector c_a and this endowment vector. By following Rader (1964) and others, however, and considering only feasible net trade vectors, the formulation here allows domestic production activities such as storage, as well as the kind of economy with small farmers considered in Coles and Hammond (1986). Note the assumption that autarky is feasible, but note too that this does *not* imply that autarky enables an agent to survive.

2.1.4. Feasible Allocations

An *allocation* $\mathbf{x} := \langle x_a \rangle_{a \in A} \in \prod_{a \in A} X_a$ is a profile of net trade vectors which are individually feasible for all agents $a \in A$. A *feasible allocation* $\mathbf{x}$ also has the property that the aggregate net trade vector $\sum_{a \in A} x_a = 0$. Note that only an exchange economy is being considered, and that free disposal is assumed only to the extent that some individual agents can dispose of goods freely.

2.1.5. Consumers' Preferences

It is also assumed that every agent $a \in A$ has a (complete and transitive) weak *preference ordering* $\succsim_a$ on the set X_a, and an associated strict preference relation $\succ_a$ which is *locally non-satiated* — i.e., for every $x_a \in X_a$ the *preferred set* $P_a(x_a) := \{ x'_a \in X_a \mid x'_a \succ_a x_a \}$ is non-empty and includes the point x_a in its closure cl $P_a(x_a)$. Note especially that there is no assumption of monotonicity, free disposal, convexity, or even continuity.

In addition to the notation $P_a(x_a)$ introduced above, let

$$U_a(x_a) := \{ x'_a \in X_a \mid x'_a \succsim_a x_a \} \quad \text{and} \quad L_a(x_a) := \{ x'_a \in X_a \mid x_a \succsim_a x'_a \}$$

denote the consumer's *upper* and *lower contour sets*, respectively.

2.1.6. The Classical Hypotheses

Some later results, however, will require standard convexity and continuity hypotheses. Indeed, say that agents' preferences *satisfy the classical hypotheses* provided that, for all $a \in A$, in addition to local non-satiation, the following are true:

(i) the set X_a of feasible net trade vectors is convex;

(ii) whenever $x'_a \succsim_a x_a$, $x''_a \succsim_a x_a$, and also $0 < \lambda \leq 1$, then $(1-\lambda)\, x'_a + \lambda\, x''_a \succsim_a x_a$, and if also $x''_a \succ_a x_a$, then $(1-\lambda)\, x'_a + \lambda\, x''_a \succ_a x_a$;

(iii) both the feasible set X_a and, for every $x_a \in X_a$, the lower contour set $L_a(x_a)$, are closed sets.

The above classical hypotheses are automatically satisfied when the set of all feasible allocations is convex, and when preferences are convex, continuous, and locally non-satiated. Of course, these latter are standard assumptions in general equilibrium theory for finite economies; indeed, (iii) above is usually supplemented by the requirement that each upper contour set $U_a(x_a)$ is also closed.

2.1.7. The Price Domain

The set of all allowable price vectors will be $\{ p \in \Re^G \mid p \neq 0 \}$. Note that negative prices are allowed because there has been no assumption of free disposal.

2.2. Equilibrium

2.2.1. The Budget, Demand, and Compensated Demand Correspondences

For each agent $a \in A$, wealth level w_a, and price vector $p \neq 0$, define the *budget set*

$$B_a(p, w_a) := \{ x \in X_a \mid p\,x \leq w_a \}$$

of feasible net trade vectors satisfying the budget constraint. Note that $B_a(p, w_a)$ is never empty when $w_a \geq 0$ because of the assumption that $0 \in X_a$.

Next define, for every $a \in A$ and $p \neq 0$, the following three demand sets:

(i) the *uncompensated demand set* is given by

$$\begin{aligned} \xi_a^U(p, w_a) &:= \{ x \in B_a(p, w_a) \mid x' \in P_a(x) \Longrightarrow p\,x' > w_a \} \\ &= \arg\max_x \{ \precsim_a \mid x \in B_a(p, w_a) \}; \end{aligned}$$

(ii) the *compensated demand set* is given by

$$\xi_a^C(p, w_a) := \{ x \in B_a(p, w_a) \mid x' \in U_a(x) \Longrightarrow p\,x' \geq w_a \}.$$

(iii) the *weak compensated demand set* is given by

$$\xi_a^W(p, w_a) := \{ x \in B_a(p, w_a) \mid x' \in P_a(x) \Longrightarrow p\,x' \geq w_a \}.$$

Evidently these definitions imply that $\xi_a^U(p, w_a) \cup \xi_a^C(p, w_a) \subset \xi_a^W(p, w_a)$. Establishing when $\xi_a^C(p, w_a) = \xi_a^U(p, w_a)$ at an equilibrium price vector p is, of course, one of the main topics of the paper. The following lemma shows that, because of local non-satiation, demands of all three kinds always exhaust the budget, and also there is in fact never any need to consider weak compensated demands, since they become equal to compensated demands.

LEMMA. *Whenever preferences are locally non-satiated, it must be true that: (i)* $x \in \xi_a^W(p, w_a) \Longrightarrow p\,x = w_a$*; (ii)* $\xi_a^W(p, w_a) = \xi_a^C(p, w_a)$.

PROOF: (i) If $p\,x < w_a$ then, since local non-satiation implies that $x \in \text{cl}\; P_a(x)$, there must also exist $x' \in P_a(x)$ with $p\,x' < w_a$, and so $x \notin \xi_a^W(p, w_a)$. Conversely, $x \in \xi_a^W(p, w_a)$ must imply that $p\,x \geq w_a$. But since $x \in \xi_a^W(p, w_a)$ implies $x \in B_a(p, w_a)$ and so $p\,x \leq w_a$, it must actually be true that $x \in \xi_a^W(p, w_a)$ implies $p\,x = w_a$.

(ii) Suppose that $\hat{x} \in \xi_a^W(p, w_a)$. For any $x' \in U_a(\hat{x})$, local non-satiation implies that $x' \in \text{cl } P_a(x')$. But $P_a(x') \subset P_a(\hat{x})$ because preferences are transitive, and so $x' \in \text{cl } P_a(\hat{x})$. Now, since $\hat{x} \in \xi_a^W(p, w_a)$ implies that $p\,x \geq w_a$ for all $x \in P_a(\hat{x})$, the same must also be true for all $x \in \text{cl } P_a(\hat{x})$, including x'. Therefore $x' \in U_a(\hat{x})$ implies $p\,x' \geq w_a$, and so $\hat{x} \in \xi_a^C(p, w_a)$. ∎

2.2.2. Compensated and Uncompensated Equilibria

An *uncompensated* (resp. *compensated*) *equilibrium* is a pair $(\mathbf{x}, p)$ consisting of a feasible allocation $\mathbf{x}$ and a price vector p such that, for all $a \in A$, both $p\,x_a = 0$ and $x_a \in \xi_a^U(p, 0)$ (resp. $\xi_a^C(p, 0)$).

An *uncompensated* (resp. *compensated*) *equilibrium with transfers* is a pair $(\mathbf{x}, p)$ consisting of a feasible allocation $\mathbf{x}$ and a price vector p such that $x_a \in \xi_a^U(p, p\,x_a)$ (resp. $\xi_a^C(p, p\,x_a)$) for all $a \in A$.[5]

[5] Honkapohja (1987) has recently used a different definition of compensated equilibrium, based on the alternative (and often used) definition

$$D_a(p, x_a^*) := \arg\min_x \{\, p\,x \mid x \in U_a(x_a^*) \,\}.$$

of the compensated demand set. Thus there is no reference whatsoever to the consumer's available wealth. Indeed, in the special case of an exchange economy in which each agent's consumption set is the non-negative orthant, according to this definition there is always a trivial equilibrium in which all endowments get thrown away and each consumer has zero consumption. This is because Honkapohja allows free disposal, however. Duffie (1988, p. 44) also uses this definition of compensated demand, in effect, but his definition of compensated equilibrium turns out to be the same as the standard one given here because he does not allow free disposal. Also, these observations only apply to Honkapohja's discussion of compensated equilibrium, and not to his other results concerning compensated demand.

3. Cheaper Points for Individuals

3.1. Arrow's Exceptional Case

Recall that the second efficiency theorem states conditions under which a Pareto efficient allocation is an uncompensated equilibrium with lump-sum redistribution — or, in a weaker version, under which such an allocation is a compensated equilibrium with lump-sum redistribution. Both Arrow (1951) and Debreu (1951) announced versions of this theorem in the same year. It seems clear that Arrow was the first to realize the nature of the problem of how to prove that a compensated equilibrium with expenditure minimization by agents is also an uncompensated equilibrium with preference maximization by agents. Indeed, in that paper he gave a famous example of an "exceptional case" in which expenditure minimization is insufficient to ensure preference maximization. On the other hand Debreu (1951) simply passed from compensated to uncompensated equilibrium without any attempt at a proof. He was apparently unaware of the possibility of any exceptional case, even though he acknowledged having seen an early version of Arrow's paper. Of course this logical flaw was later set right in his succeeding works such as Debreu (1954, 1959, 1962).

3.2. Interiority

3.2.1. Interiority of a Particular Allocation

In fact Arrow's paper was presented at the same symposium as Kuhn and Tucker's (1951) classic work on non-linear programming. They also gave an example in which a constrained maximum could not be supported by shadow prices — something which is impossible for linear programs in finite dimensional spaces, because for those the well known duality theorem applies. The simplest instance of a constraint qualification is Slater's condition, requiring that a concave program have a point in the interior of its feasible set.

Arrow gave two alternative conditions which rule out exceptional cases. The first of these is a direct parallel of the Slater constraint qualification. This is the "interiority assumption" which simply assumes that the Pareto efficient allocation being considered gives each agent a net trade vector in the interior of his feasible set. For the case of convex feasible sets and continuous preferences, this is certainly enough to convert a compensated equilibrium with lump-sum redistribution into an uncompensated equilibrium with identical lump-sum redistribution, and to make the second efficiency theorem true.

3.2.2. Interiority of Initial Endowments

A somewhat different interiority assumption was also used in the first existence theorem of Arrow and Debreu (1954), and in immediately succeeding works such as Gale (1955), Nikaido (1956), and Debreu (1959). For the second efficiency theorem, the interiority assumption could refer to the particular Pareto efficient allocation whose competitive properties are being demonstrated. For an existence theorem, however, it makes no sense to refer to the equilibrium allocation until we know whether one exists. Nor should we assume that a compensated equilibrium allocation is an interior allocation, since there are many examples where it will not be — for instance, if some individuals are unconcerned about the consumption of some goods, compensated equilibrium at strictly positive prices implies that these individuals are at the lower boundaries of their consumption sets. For this reason, then, it seemed more natural to assume that all individuals had endowment vectors in the interior of their consumption sets — or, somewhat more generally, that the zero net trade vector was in the interior of each individual's set of feasible net trades. Since feasibility was nearly always assumed to imply survival, this was actually a kind of "strict survival" assumption. But in fact, as remarked above, there is actually no need to assume that feasibility implies survival. After making this survival assumption, as well as the usual assumptions that individuals' feasible sets are convex and their preferences are continuous, it becomes quite straightforward to show that any compensated equilibrium is actually an uncompensated equilibrium.

Of course Debreu (1959) and many successors prefer to use the assumption that endowments are interior in order to demonstrate upper hemi-continuity of the uncompensated rather than of the compensated or of the quasi demand correspondence, and then go on to prove existence of an uncompensated equilibrium by applying a fixed point theorem directly to the aggregate excess uncompensated demand correspondence. This approach, however, tends to obscure what role the objectionable interiority assumption plays in the proof of existence. More seriously, it may also make existence proofs harder than necessary, especially in "non-classical" economies such as those where there is a continuum of consumers who may have non-convex feasible sets. For such economies it is often easier to follow the techniques of Khan and Yamazaki (1981) and Yamazaki (1978, 1981) in order to prove existence of compensated equilibrium first, and then use their results or those of this paper in order to prove that a compensated equilibrium is also an uncompensated equilibrium.

The absurdity of this assumption that each consumer's endowments are interior was immediately apparent. Early writers would typically apologize for it and would sometimes seek to weaken it.[6] After all, in a model of competitive equilibrium international trade in the world economy, non-traded goods such as local construction and other labour services have to be differentiated by the country in which they are supplied. Then the interiority assumption requires each individual to be able to supply every non-traded good in every country of the world simultaneously. Not even internationally mobile professors of economics have this capability! Indeed, once commodities become distinguished by time and geographical location, as they should be, interiority requires all agents to be both omnipresent and immortal. This kind of consideration surely motivated Nikaido's (1957) ingenious relaxation of the interiority assumption. But he had to assume instead that all agents have the non-negative orthant as their consumption set, that there is a positive total endowment of every commodity, and also that preferences are strictly monotone — in particular, that every good is strictly desirable.[7] The last assumption is hardly more acceptable than interiority when one considers goods differentiated by location. Chinese tea is wonderful almost anywhere at almost any time, but all the tea in China only benefits those who are actually there.

3.3. Cheaper Feasible Points

3.3.1. Introduction

Despite its evident absurdity, the interiority assumption remains the one which is most frequently encountered in the general equilibrium literature. Yet alternatives have been proposed, starting with a second assumption in Arrow (1951) which was weaker than the interiority condition he had used first. Whereas interiority requires the agent to be able to supply a little of every good, Arrow's second assumption is that every agent can supply

[6] Kenneth Arrow tells me that at first both he and Debreu, in independent unpublished work, had overlooked the need for some additional assumption such as interiority. Later they held up the publication of their joint paper Arrow and Debreu (1954) while they formulated a less unsatisfactory alternative to interiority. This alternative is discussed in Section 5.1 below.

[7] Actually, despite Nikaido's contribution, many subsequent papers in general equilibrium theory seem to combine non-negative orthant consumption sets, strictly monotone preferences, and the interiority assumption which he showed to be unnecessary in the presence of the other two assumptions.

a little of at least one good having a positive price in the compensated equilibium price system. Unfortunately, however, this assumption has the obvious defect that it can only be checked after the compensated equilibium price system has become known. Or at least it is necessary to know which goods have zero or even negative prices, and which goods have positive prices, in this system.

3.3.2. Cheaper Points for Non-Convex Feasible Sets

The usual demonstration that the existence of a cheaper feasible point for an individual converts a compensated into an uncompensated equilibrium for that individual depends upon the feasible set being convex. To generalize this to non-convex feasible sets requires an additional assumption. When some goods are indivisible, it has been common to assume that divisible goods are "overridingly desirable," meaning that any commodity bundle with indivisible goods can be improved by moving to another with only divisible goods. This assumption is discussed below in Section 3.3.3. An attempt to generalize it follows.

First, let $\mathrm{co}\, Y$ and $\mathrm{cl}\ Y$ denote respectively the convex hull and the closure of any set $Y \subset \Re^G$. Then, for every $a \in A$ and $x_a \in X_a$, let:

$$K_a(x_a) := \mathrm{co}\{\, x'_a \in \Re^G \mid \exists \tilde{x}_a \in X_a \ \&\ \exists \lambda > 0 : x'_a = x_a + \lambda\,(\tilde{x}_a - x_a)\,\};$$
$$K_a^{\succ}(x_a) := \mathrm{co}\{\, x'_a \in \Re^G \mid \exists \tilde{x}_a \in P_a(x_a) \ \&\ \exists \lambda > 0 : x'_a = x_a + \lambda\,(\tilde{x}_a - x_a)\,\}.$$

Thus $K_a(x_a)$ denotes the smallest convex cone with vertex x_a which is generated by the set X_a of feasible net trade vectors, while $K_a^{\succ}(x_a)$ denotes the corresponding cone which is generated by the set $P_a(x_a)$ of net trade vectors which are strictly preferred to x_a. Because $x_a \in X_a$, note that $K_a(x_a)$ is a *pointed cone* containing its own vertex x_a. On the other hand $x_a \notin P_a(x_a)$, and so it may well be true that $x_a \notin K_a^{\succ}(x_a)$, in which case the cone $K_a^{\succ}(x_a)$ is *unpointed* — indeed, this will always be the case when agent a has convex preferences.

Assumption 1. *For every agent $a \in A$ and net trade vector $x_a \in X_a$, if $x'_a \in K_a^{\succ}(x_a)$ and $\bar{x}_a \in K_a(x_a) \setminus \mathrm{cl}\ K_a^{\succ}(x_a)$, then the (relatively) open line segment $L := (x'_a, \bar{x}_a) \subset K_a(x_a)$ of points lying strictly between x'_a and $\bar{x}_a$ must have a non-empty intersection $L \cap K_a^{\succ}(x_a)$ with the convex cone $K_a^{\succ}(x_a)$.*[8]

[8] As stated here, Assumption 1 is strictly weaker than the requirement that the convex cone $K_a^{\succ}(x_a)$ should be open relative to the convex cone $K_a(x_a)$ — i.e., that $K_a^{\succ}(x_a) = O \cap \bar{K}_a(x_a)$ for some set O that is open in $\Re^G$.

Note that often $K_a(x_a)$ will be the whole space $\Re^G$, but this still makes Assumption 1 meaningful — in this case it simply requires $K_a^{\succ}(x_a)$ to be an open set. The convex cone $K_a^{\succ}(x_a)$ may also be the whole space $\Re^G$ if preferences are non-convex, in which case Assumption 1 is automatically satisfied at x_a.

Assumption 1 is perhaps not as easy to check as it should be. Nevertheless, it gives what is needed by ruling out such troublesome examples as those discussed in Broome (1972), Mas-Colell (1977), and Khan and Yamazaki (1981). In addition, it generalizes the more familiar condition which is the hypothesis of the following:

LEMMA. *Suppose that the feasible set X_a is convex and that the lower contour set $L_a(x_a)$ is closed for each $x_a \in X_a$. Then Assumption 1 is satisfied.*

PROOF: Let x_a be any net trade vector in X_a. Suppose that $x'_a \in K_a^{\succ}(x_a)$ and that $\bar{x}_a \in K_a(x_a) \setminus \text{cl}\, K_a^{\succ}(x_a)$. Then $L := (x'_a, \bar{x}_a) \subset K_a(x_a)$.

Because $x'_a \in K_a^{\succ}(x_a)$ and $\bar{x}_a \in K_a(x_a)$, there must exist finite sets of net trade vectors $x^j \in P_a(x_a), y^i \in X_a$ and of associated positive scalars λ^j, μ^i ($j = 1$ to J; $i = 1$ to I) such that

$$x'_a - x_a = \sum\nolimits_{j=1}^{J} \lambda^j \,(x^j - x_a) \quad \text{and} \quad \bar{x}_a - x_a = \sum\nolimits_{i=1}^{I} \mu^i \,(y^i - x_a).$$

Let $\mu := \sum_{i=1}^{I} \mu^i$ and $y := \sum_{i=1}^{I} (\mu^i/\mu)\, y^i$. Then $y \in X_a$ because X_a is convex. Also $\bar{x}_a - x_a = \mu\,(y - x_a)$.

Now $x^j \in P_a(x_a)$ and $y \in X_a$ while X_a is convex and the lower contour set $L_a(x_a)$ is closed. So, for each $j = 1$ to J there must exist an ϵ^j with $0 < \epsilon^j \leq 1$ such that $x^j + \epsilon\,(y - x^j) \in P_a(x_a)$ whenever $0 < \epsilon < \epsilon^j$. Then, however, for the vector $x^\epsilon := x'_a + \epsilon\,(\bar{x}_a - x'_a) \in L$ it must be true that

$$\begin{aligned} x^\epsilon - x_a &= (1-\epsilon)\,(x'_a - x_a) + \epsilon\,(\bar{x}_a - x_a) \\ &= (1-\epsilon) \sum\nolimits_{j=1}^{J} \lambda^j \,(x^j - x_a) + \epsilon\,\mu\,(y - x_a) \\ &= \sum\nolimits_{j=1}^{J} \frac{(1-\epsilon)\,\lambda^j}{1-\delta}\,[x^j + \delta\,(y - x^j) - x_a] \end{aligned}$$

where $\epsilon\,\mu\,(1-\delta) = (1-\epsilon)\,\delta \sum_{j=1}^{J} \lambda^j$ and so $\delta = \epsilon\,\mu/[\epsilon\,\mu + (1-\epsilon)\sum_{j=1}^{J} \lambda^j]$. This implies that $x^\epsilon \in K_a^{\succ}(x_a)$ as long as $0 < \delta < \epsilon^j$ and so $x^j + \delta\,(y - x^j) \in P_a(x_a)$ for $j = 1$ to J. This requires that

$$0 < \epsilon < \min_j \left\{ \frac{\epsilon^j \sum_{i=1}^{J} \lambda^i}{\mu\,(1-\epsilon^j) + \sum_{i=1}^{J} \lambda^i} \right\}.$$

Since the interval of such values of ϵ is certainly non-empty, so is $L \cap K_a^{\succ}(x_a)$. ∎

3.3.3. Overriding Desirability of Divisible Goods

Consider an economy with just one divisible and one indivisible good. Indeed, suppose that $X_a = \Re_+ \times Z_+$ where Z_+ denotes the set of non-negative integers, representing quantities of the indivisible good. Suppose too that preferences are *strictly monotone* in the sense that, if $x_a = (y_a, z_a)$ and $x'_a = (y'_a, z'_a)$ are both members of X_a with $y'_a \geq y_a$ and $z'_a \geq z_a$, then $x'_a \succsim_a x_a$, with strict preference unless both $y'_a = y_a$ and $z'_a = z_a$.

Next, say that *divisible goods are overridingly desirable* if, whenever $x_a = (y_a, z_a)$ and $x'_a = (y'_a, z'_a)$ are both members of X_a, then there exists $\hat{y}_a \in \Re_+$ such that $(\hat{y}_a, z'_a) \succ_a (y_a, z_a)$.[9] Thus, even though z'_a may involve much less of the indivisible good than z_a does, the move from z_a to z'_a can always be outweighed by a move from y_a to $\hat{y}_a$ in the divisible good.

Assumption 1 seems at first to be a significant weakening of overriding desirability. When some goods are indivisible, however, Assumption 1 often turns out to be no weaker than the assumption of overriding desirability, as is shown by the following.

Lemma. *Under the hypotheses of the first paragraph of this Section, Assumption 1 implies that divisible goods are overridingly desirable.*

Proof: Suppose that (y_a, z_a) is some member of X_a with $z_a > 0$. Then the hypotheses imply that the convex cone $K_a(y_a, z_a)$ with vertex (y_a, z_a) which is generated by the feasible set $X_a = \Re_+ \times Z_+$ is: either (i) the whole of $\Re^2$, in case $y_a > 0$; or (ii) just the half-space $\{ (y, z) \in \Re^2 \mid y \geq 0 \}$, in case $y_a = 0$.

Suppose it were true that $(y_a, z_a) \succsim_a (\tilde{y}_a, z_a - 1)$ whenever $\tilde{y}_a \geq 0$. Then the convex cone $K_a^{\succ}(y_a, z_a)$ with vertex (y_a, z_a) which is generated by the preference set $P_a(y_a, z_a)$ would be a subset of $\{ (y, z) \in \Re^2 \mid z \geq z_a \}$. Consider now any $(y^*, z^*) \in \Re^2$ satisfying $y^* > y_a$ and $z^* = z_a$. By strict monotonicity of preferences one has $(y^*, z^*) \succ_a (y_a, z_a)$, implying that (y^*, z^*) would not only belong to $K_a^{\succ}(y_a, z_a)$ but also to the boundary of this

[9] This assumption appears to have originated with Broome (1972). It has since been used by, amongst others, Mas-Colell (1977), Khan and Yamazaki (1981), and Hammond, Kaneko and Wooders (1989). Henry (1970) earlier formulated a condition that the divisible good be necessary for survival. However, this condition is stated in a way which appears to be inconsistent with his other assumptions. For, in the notation used here, it requires that, whenever $x_a = (y_a, z_a)$ and $x'_a = (y'_a, z'_a)$ are both members of X_a, then there should exist $\hat{y}_a \in \Re_+$ such that $(y_a, z_a) \succ_a (\hat{y}_a, z'_a)$. This can never hold, of course, if $(y_a, z_a) = (0, 0)$, the worst point in X_a. If instead the condition is only required to hold for those $x_a = (y_a, z_a) \in \Re_+ \times Z_+$ which satisfy $y_a > 0$, then there is no inconsistency, but the assumption becomes quite different from overriding desirability.

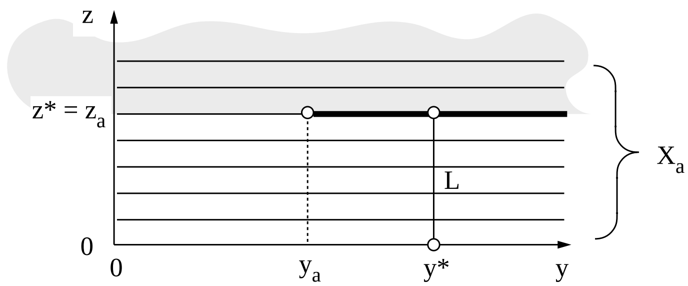

Figure 1

set — which is (partly) illustrated as the shaded set in Figure 1. So, if L is the open line segment with end-points (y^*, z^*) and $(y^*, 0)$, then $L \cap K_a^{\succ}(y_a, z_a)$ would be empty, thereby contradicting Assumption 1.

Therefore, for every $(y_a, z_a) \in X_a$ with $z_a > 0$, there must exist $\tilde{y}_a \geq 0$ for which $(\tilde{y}_a, z_a - 1) \succ_a (y_a, z_a)$. Because preferences are transitive, an easy argument by induction on the integer z_a then establishes that the one divisible good is overridingly desirable. ∎

Although this lemma has been proved only for the case on one divisible and one indivisible good, it is clear that the argument can be greatly generalized to allow many divisible and indivisible goods. The issue of whether there is an adequate weaker assumption than overriding desirability of divisible goods will be taken up in Section 3.3.5 below.

3.3.4. The Cheaper Point Theorem

PROPOSITION. *Suppose that $\hat{x}_a \in \xi^C(p, w_a)$ is any compensated equilibrium for agent a at the price vector $p \neq 0$. If Assumption 1 is satisfied, and if there also exists $\bar{x}_a \in X_a$ with $p\,\bar{x}_a < w_a$, then $\hat{x}_a \in \xi^U(p, w_a)$ — i.e., $\hat{x}_a$ is an uncompensated equilibrium.*

PROOF: Suppose that $\hat{x}_a \in \xi^C(p, w_a) \setminus \xi^U(p, w_a)$. By Lemma 2.2.1, $p\,\hat{x}_a = w_a$. Because $\hat{x}_a \notin \xi^U(p, w_a)$, there exists $x'_a \in P_a(\hat{x}_a)$ such that $p\,x'_a \leq w_a$. Since $\hat{x}_a \in \xi^C(p, w_a)$ it follows that $x_a \in U_a(\hat{x}_a)$ implies $p\,x_a \geq p\,\hat{x}_a = w_a$ and so in fact $p\,x'_a = w_a$. Also, one must have $p\,x_a \geq w_a$ for all $x_a \in K_a^{\succ}(\hat{x}_a)$ and so for all $x_a \in \text{cl } K_a^{\succ}(\hat{x}_a)$.

Since $p\,\bar{x}_a < w_a = p\,x'_a = p\,\hat{x}_a$, it must be true that $\bar{x}_a \in K_a(\hat{x}_a) \setminus \text{cl } K_a^{\succ}(\hat{x}_a)$. Let L denote the non-empty open line segment $(x'_a, \bar{x}_a) \subset K_a(\hat{x}_a)$. Suppose now that the set $L \cap K_a^{\succ}(\hat{x}_a)$ were non-empty, in which case there would exist an ϵ for which $0 < \epsilon < 1$ and

$$x_a^\epsilon := x'_a + \epsilon\,(\bar{x}_a - x'_a) \in K_a^{\succ}(\hat{x}_a).$$

Then there would be a finite collection of points $x_a^j \in P_a(\hat{x}_a)$ and of associated positive scalars λ_j ($j = 1$ to J) such that $x_a^\epsilon - \hat{x}_a = \sum_{j=1}^J \lambda_j (x_a^j - \hat{x}_a)$. Because $p\, x_a' = w_a = p\, \hat{x}_a$ and $p\, \bar{x}_a < p\, x_a'$, this would imply that

$$\sum_{j=1}^{J} \lambda_j\, p\, (x_a^j - \hat{x}_a) = p\, (x_a^\epsilon - \hat{x}_a) = p\, (x_a' - \hat{x}_a) + \epsilon\, p\, (\bar{x}_a - x_a') < 0.$$

Since each scalar λ_j is positive, this would imply that there must be at least one j for which $p\, (x_a^j - \hat{x}_a) < 0$. But $x_a^j \in P_a(\hat{x}_a)$ and so this contradicts the hypothesis that $\hat{x}_a \in \xi^C(p, w_a)$. Therefore $L \cap K_a^{\succ}(\hat{x}_a)$ must be empty after all.

Thus, if $\hat{x}_a \in \xi^C(p, w_a) \setminus \xi^U(p, w_a)$, then Assumption 1 is violated. Conversely, if Assumption 1 is satisfied, then $\xi^C(p, w_a) = \xi^U(p, w_a)$. ∎

3.3.5. The Necessity of Assumption 1

Though Assumption 1 has some undesirable implications which were discussed in Section 3.3.3, the following result shows that there is a sense in which it is indispensable.

Proposition. *Suppose there exist net trade vectors $\hat{x}_a \in X_a$, $x_a' \in K_a^{\succ}(\hat{x}_a)$ and $\bar{x}_a \in K_a(\hat{x}_a) \setminus \text{cl}\ K_a^{\succ}(\hat{x}_a)$ for which the open line segment $L := (x_a', \bar{x}_a)$ is disjoint from the convex cone $K_a^{\succ}(\hat{x}_a)$. Then there exists a price vector $p \neq 0$ and a wealth level $w_a := p\, \hat{x}_a$ such that $\hat{x}_a \in \xi_a^C(p, w_a)$ and yet $\hat{x}_a \notin \xi_a^U(p, w_a)$, even though $p\, \bar{x}_a < w_a$ — i.e., the compensated equilibrium net trade vector $\hat{x}_a$ is not an uncompensated equilibrium, even though X_a has a cheaper point at the equilibrium price vector p.*

Proof: (1) The hypotheses of the proposition imply that the point $\bar{x}$ has an open and convex neighbourhood V which is disjoint from cl $K_a^{\succ}(\hat{x}_a)$. Let K^0 be the smallest unpointed convex cone with vertex $\hat{x}_a$ which contains $L \cup V$. Evidently K^0 is non-empty, open, convex, and also disjoint from the convex cone $K_a^{\succ}(\hat{x}_a)$ with the same vertex $\hat{x}_a$. So there must be a price vector $p \neq 0$ and an associated hyperplane $p\, x_a = \alpha$ which separates the two convex sets K^0 and $K_a^{\succ}(\hat{x}_a)$ in the sense that $x \in K^0 \Longrightarrow p\, x \leq \alpha$ and $x \in K_a^{\succ}(\hat{x}_a) \Longrightarrow p\, x \geq \alpha$.

(2) Let $w_a := \alpha$. In fact, since the two points $\hat{x}_a$ and x_a' both lie in the intersection of the respective closures of the two convex cones $K_a^{\succ}(\hat{x}_a)$ and K^0, both these points must actually lie in the separating hyperplane, and so $p\, \hat{x}_a = p\, x_a' = \alpha = w_a$.

(3) Moreover, since K^0 is open and $x \in K^0 \Longrightarrow p\, x \leq \alpha$, it is actually true that $x \in K^0 \Longrightarrow p\, x < \alpha$. In particular, since $\frac{1}{2}\, x_a' + \frac{1}{2}\, \bar{x}_a \in L^0 \subset K^0$ and $p\, x_a' = \alpha$, one must have $p\, (\frac{1}{2}\, x_a' + \frac{1}{2}\, \bar{x}_a) < p\, x_a'$ and so $p\, \bar{x}_a < p\, x_a' = w_a$.

(4) In addition, since $P_a(\hat{x}_a) \subset K_a^{\succ}(\hat{x}_a)$, it follows that $x_a \in P_a(\hat{x}_a)$ implies $p\, x_a \geq w_a$.

(5) Because (2) implies that $p\,\hat{x}_a = w_a$, and also because of local non-satiation and Lemma 2.2.1, (4) shows that $\hat{x}_a \in \xi_a^C(p, w_a)$.

(6) Finally, because $x'_a \in K_a^{\succ}(\hat{x}_a)$, there exists a finite collection of net trade vectors $x_a^j \in P_a(\hat{x}_a)$ and of associated positive scalars λ^j ($j = 1$ to J) for which $x'_a - \hat{x}_a = \sum_{j=1}^J \lambda^j\,(x_a^j - \hat{x}_a)$. But, by (2) and (4), $x_a \in P_a(\hat{x}_a)$ implies $p\,x_a \geq w_a = p\,\hat{x}_a$, and so $p\,x_a^j \geq p\,\hat{x}_a$ for $j = 1$ to J.

(7) Also, since (2) implies that $p\,\hat{x}_a = p\,x'_a$, it follows from (6) that

$$0 = p\,(x'_a - \hat{x}_a) = \sum_{j=1}^J \lambda^j\,p\,(x_a^j - \hat{x}_a) \geq 0$$

and so $\sum_{j=1}^J \lambda^j\,p\,(x_a^j - \hat{x}_a) = 0$.

(8) Since $\lambda^j > 0$ and, by (6), $p\,x_a^j \geq p\,\hat{x}_a$ ($j = 1$ to J), (2) and (7) together imply that $p\,x_a^j = p\,\hat{x}_a = w_a$ for $j = 1$ to J. But (6) implies that $x_a^j \in P_a(\hat{x}_a)$ ($j = 1$ to J), and so $\hat{x}_a \notin \xi_a^U(p, w_a)$. ∎

3.4. *Aggregate Interiority*

3.4.1. *The Interiority Assumption*

In a finite economy, in order for some individual to have a cheaper feasible net trade, it is necessary that society as a whole should have some cheaper point. So assume:

ASSUMPTION 2. *The origin $0 \in \Re^G$ belongs to the interior of K_A, the convex hull of the set $X_A := \sum_{a \in A} X_a$.*

This assumption can be motivated as follows. Back in Section 2.1.3 it was assumed that $0 \in X_a$ for all $a \in A$. So if Assumption 2 is false, it can only be because 0 is a boundary point of K_A. But then there would be a hyperplane $p\,x = 0$ through 0 which supports the convex set K_A in the sense that $p\,x \geq 0$ whenever $x \in K_A$. Note how $X_a \subset X_A \subset K_A$ (all $a \in A$) because $0 \in X_{a'}$ for all $a' \in A \setminus \{a\}$. So, for all $a \in A$, it would follow that $p\,x \geq 0$ whenever $x \in X_a$. Thus almost no agent in the economy can trade in the open half-space H of directions x satisfying $p\,x < 0$. If almost nobody can trade in any such direction, net demands for vectors in the opposite half-space $-H$ are completely ineffective. They are like asking for the moon — or at least, for a return trip there this year (1991). It therefore makes sense to restrict attention to the proper linear subspace of net trade vectors satisfying $p\,x = 0$. Repeating this argument as many times as necessary results in a commodity space which is the linear space spanned by the relative interior of the convex set K_A. For this commodity space, of course, Assumption 2 is satisfied by construction.

3.4.2. Cheaper Points for Some Individuals

LEMMA. *Under Assumption 2, for every feasible allocation* $\mathbf{x}$ *and every price vector* $p \neq 0$*, the set*

$$\{ a \in A \mid \exists \bar{x}_a \in X_a \text{ such that } p\,\bar{x}_a < p\,x_a \}$$

of individuals with cheaper points is non-empty.

PROOF: Assumption 2 implies that for every price vector $p \neq 0$ there exists $\tilde{x} \in K_A$ such that $p\,\tilde{x} < 0$. But then there must exist $\bar{x} \in \sum_{a\in A} X_a$ for which $p\,\bar{x} < 0$. Let $\bar{x} = \sum_{a\in A} \bar{x}_a$ where $\bar{x}_a \in X_a$ for all $a \in A$. Then, since aggregate feasibility implies that $\sum_{a\in A} x_a = 0$, it must be true that

$$\sum_{a\in A} p\,(\bar{x}_a - x_a) = \sum_{a\in A} p\,\bar{x}_a = p\,\bar{x} < 0.$$

This is only possible, however, if $p\,\bar{x}_a < p\,x_a$ for at least one $a \in A$. ∎

4. Non-Oligarchic Allocations

4.1. Definitions

Suppose that $\mathbf{x}$ is a feasible allocation. Let C be any proper subset of A. Say that "$A \setminus C$ may strongly improve the condition of C in $\mathbf{x}$" if there exists an alternative feasible allocation $\mathbf{x}'$ such that $x'_a \succ_a x_a$ for all $a \in C$. And, as in Spivak (1978), say that "$A \setminus C$ may improve the condition of C in $\mathbf{x}$" if there exists an alternative feasible allocation $\mathbf{x}'$ such that $x'_a \succsim_a x_a$ for all $a \in C$, and $x'_{a^*} \succ_{a^*} x_{a^*}$ for some $a^* \in C$.

Next, C is said to be a *weak oligarchy* at $\mathbf{x}$ if $A \setminus C$ cannot strongly improve the condition of C in $\mathbf{x}$. Equivalently, C is a weak oligarchy at $\mathbf{x}$ if there is no other feasible allocation $\mathbf{x}'$ such that $x'_a \succ_a x_a$ for all $a \in C$ — i.e., if

$$0 \notin \sum_{a\in C} P_a(x_a) + \sum_{a\in A\setminus C} X_a.$$

And C is said to be a *strong oligarchy* at $\mathbf{x}$ if $A \setminus C$ cannot improve the condition of C in $\mathbf{x}$. Equivalently, C is a strong oligarchy at $\mathbf{x}$ if there is no other feasible allocation $\mathbf{x}'$ such that $x'_a \succsim_a x_a$ for all $a \in C$, and $x'_{a^*} \succ_{a^*} x_{a^*}$ for some $a^* \in C$ — i.e., if

$$0 \notin P_{a^*}(x_{a^*}) + \sum_{a \in C \setminus \{a^*\}} U_a(x_a) + \sum_{a \in A \setminus C} X_a$$

for all $a^* \in C$.

The last definitions receive their name because an omnipotent oligarchy would presumably choose a feasible allocation with the (strong oligarchy) property that no alternative feasible allocation could make some of its members better off without simultaneously making some other members worse off. Or it would at least choose a feasible allocation with the (weak oligarchy) property that no alternative feasible allocation could make all of its members better off simultaneously.

Conversely, the feasible allocation $\mathbf{x}$ is said to be *strongly* (resp. weakly) *non-oligarchic* if there is no weak (resp. strong) oligarchy at $\mathbf{x}$. Thus, $\mathbf{x}$ is strongly non-oligarchic if and only if, for every partition $A_1 \cup A_2$ of A into two disjoint non-empty sets A_1, A_2, there exists another feasible allocation $\mathbf{x}'$ such that $x'_a \succ_a x_a$ for all $a \in A_1$ — i.e., if

$$0 \in \sum_{a \in A_1} P_a(x_a) + \sum_{a \in A_2} X_a.$$

And $\mathbf{x}$ is weakly non-oligarchic if and only if, for every partition $A_1 \cup A_2$ of A into two disjoint non-empty sets A_1, A_2, there exists another feasible allocation $\mathbf{x}'$ such that $x'_a \succsim_a x_a$ for all $a \in A_1$, and $x'_{a^*} \succ_{a^*} x_{a^*}$ for some $a^* \in A_1$ — i.e., if there exists $a^* \in A_1$ such that

$$0 \in P_{a^*}(x_{a^*}) + \sum_{a \in A_1 \setminus \{a^*\}} U_a(x_a) + \sum_{a \in A_2} X_a.$$

Of course, any strongly non-oligarchic allocation is weakly non-oligarchic.

In a non-oligarchic allocation, by contrast to an oligarchic allocation, every coalition which is a proper subset of A, the set of all agents, does have an improvement available, provided it can suitably exploit the resources of the complementary coalition. The force of the non-oligarchy assumption is that the allocation leaves every coalition with some resources which the complementary coalition would like to exploit if it could. Provided that some agents are in uncompensated equilibrium, this will ensure that the other agents do have cheaper points, as Prop. 4.3 below demonstrates.

4.2. Gale's Linear Exchange Model

Gale (1957, 1976) considered a linear exchange model in which each agent a has a feasible set $X_a = \{ x_a \in \Re^G \mid x_a \geqq \underline{x}_a \}$ of net trades, and a preference ordering $\succsim_a$ represented by the linear utility function $V_a(x_a) := v_a\, x_a = \sum_{g \in G} v_{ag}\, x_{ag}$ for some semi-positive vector v_a of coefficients. Typically the minimum net trade vector $\underline{x}_a$ of X_a is taken to be $-\omega_a$, where $\omega_a \in \Re^G_+$ is agent a's fixed endowment vector. Then a's consumption set $X_a + \{\omega_a\}$ is the non-negative orthant $\Re^G_+$. But there is no need here for this specific assumption.

In this linear exchange economy, the allocation $\mathbf{x}$ is said to be *reducible* (Eaves, 1976) if there exist two partitions $A = A_1 \cup A_2$ and $G = G_1 \cup G_2$ into two non-empty subsets of both the set of agents A and the set of goods G with the property that

$$v_{ag} = 0 \quad (a \in A_1;\ g \in G_1); \qquad x_{ag} - \underline{x}_{ag} = 0 \quad (a \in A_2;\ g \in G_2).$$

Thus the only goods desired by the members of A_1 are in the set G_2, and none of these can be supplied by the members of A_2. Consequently, a Pareto efficient allocation in this economy is reducible only if there exists an oligarchy (A_1) which cannot be made better off even with the resources of the complementary coalition (which are goods in G_1). Notice that the allocation $\mathbf{x}$ is reducible only if the matrix M defined by

$$M_{aa'} := \sum_{g \in G} v_{ag}\,(x_{a'g} - \underline{x}_{a'g}) \quad (\text{all } a, a' \in A)$$

is reducible in the sense of Gantmacher (1953, 1959) and Gale (1960) — i.e., there is a partition $A_1 \cup A_2$ of A into two non-empty subsets such that $M_{aa'} = 0$ for all $a \in A_1$ and all $a' \in A_2$. Of course, the element $M_{aa'}$ of this matrix represents the utility to a of agent a''s resources.

The converse statements are also true in this special economy. That is, if the proper subset C of A is an oligarchy at the allocation $\mathbf{x}$, then for all $a \in C$, $a' \in A \setminus C$ and $g \in G$ it must be true that

$$v_{ag} > 0 \Longrightarrow x_{a'g} - \underline{x}_{a'g} = 0.$$

This is the condition which Gale (1976) calls "self-sufficiency," although that term could perhaps be applied better to feasible allocations rather than to ways of making a coalition better off. The same condition holds, of course, if the matrix M is reducible, with an independent subset C. In either case, define

$$G_2 := \{\, g \in G \mid x_{ag} - \underline{x}_{ag} = 0 \quad (\text{all } a \in A \setminus C)\,\}.$$

Note that G_2 is certainly non-empty because it contains any good g for which there exists some $a \in C$ with $v_{ag} > 0$. This construction shows that the economy is reducible except in the uninteresting special case when $x_a = \underline{x}_a$ for all $a \in A \setminus C$, so that the coalition $A \setminus C$ has no resources at all available for redistribution to the members of C.

Thus, for the special case of the linear exchange economy, the non-oligarchy condition is very closely related to irreducibility of the matrix M defined above.

4.3. Convexified Non-Oligarchic Allocations

In fact only the convex combinations of potential changes turns out to matter. Accordingly, the proper subset C of A is said to be a *convexified oligarchy* at the feasible allocation $\mathbf{x}$ if, for every agent $a^* \in C$, one has

$$0 \notin \text{co} \left[P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in C \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A \setminus C} X_a \right].$$

The feasible allocation $\mathbf{x}$ is said to be *convexified non-oligarchic* if, for every partition $A_1 \cup A_2$ of A into two disjoint disjoint non-empty sets A_1, A_2, neither subset is a convexified oligarchy — i.e., if there exists $a^* \in A_1$ such that

$$0 \in \text{co} \left[P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in A_1 \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A_2} X_a \right].$$

Under the classical hypotheses of Section 2.1.6, any convexified non-oligarchic allocation $\mathbf{x}$ is obviously non-oligarchic. So "convexified" non-oligarchic allocations are only more general if the usual convexity hypotheses are violated. The following result is proved in Section 7.1:

PROPOSITION. *Under Assumptions 1 and 2, any convexified non-oligarchic allocation* $\mathbf{x}$ *which is a compensated equilibrium with transfers at some price vector* $p \neq 0$ *must be an uncompensated equilibrium with transfers at the same price vector* p.

4.4. *Agents outside Convexified Oligarchies Are at Cheapest Points*

Following Spivak (1978)'s proof of a similar result for non-oligarchic allocations, one can show that an allocation must be convexified non-oligarchic if every agent is to have a cheaper point at every possible compensated equilibrium price vector. Indeed, the following is proved in Section 7.2:

PROPOSITION. *Let $\hat{\mathbf{x}}$ be a feasible allocation at which the coalition C is a convexified oligarchy. Then there exists a price vector $p \neq 0$ such that $(\hat{\mathbf{x}}, p)$ is a compensated equilibrium with transfers in which all agents outside C are at cheapest points — i.e., for all $a \in A \setminus C$, it must be true that $x_a \in X_a \Longrightarrow p\, x_a \geq p\, \hat{x}_a$.*

4.5. *Two Instructive Examples*

So far I have considered necessary and sufficient conditions for a particular compensated equilibrium allocation to have the property that each agent has a cheaper point at the equilibrium prices, thus assuring (under Assumptions 1 and 2) that the equilibrium is also uncompensated. Obviously, for any compensated equilibrium to be an uncompensated equilibrium, and so for the existence of some uncompensated equilibrium, it would be sufficient for every feasible allocation to be convexified non-oligarchic. Yet this condition can never be satisfied, since there will always be extreme allocations that are oligarchic or even dictatorial. Instead one might think of assuming that at least every feasible allocation in which no agent is worse off than under autarky should be convexified non-oligarchic. This latter sufficient condition turns out to be unnecessarily strong, however, since it takes no account of the fact that the only relevant compensated equilibrium prices are those giving rise to a budget hyperplane for each agent which passes through the autarky allocation. An example to show this will now be provided.

Figure 2 illustrates a "clipped" Edgeworth box exchange economy. It is presumed that there are two perfectly complementary commodities — nuts and bolts, with quantities indicated by n and b respectively. There are also two agents labelled mnemonically B and N whose initial endowments are $(\bar{b}_B, \bar{n}_B) = (6, 0)$ and $(\bar{b}_N, \bar{n}_N) = (0, 4)$ respectively. Both agents have the obvious utility function $u(b, n) \equiv \min\{\, b, n \,\}$. For some reason which I shall not try to justify, agent B's feasible set is assumed to be not the whole of $\Re^2_+$, but only that

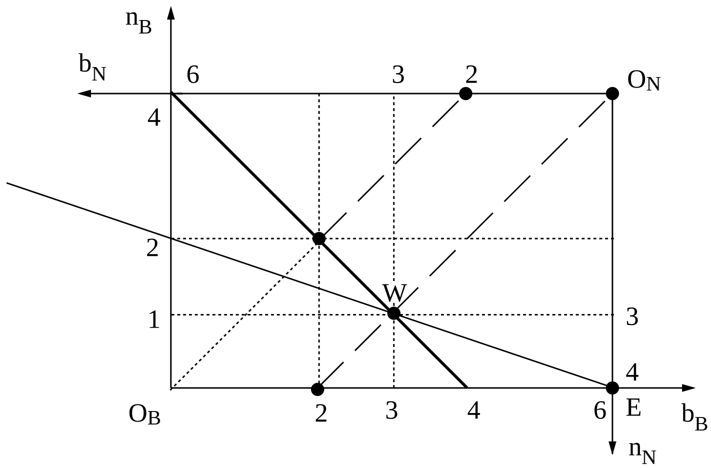

Figure 2

part of the non-negative orthant satisfying the additional constraint $b_B + n_B \geq 4$. This explains why the Edgeworth box has its corner clipped off, in effect.

Because there are more bolts than nuts, B has no market power. In fact the unique (Walrasian) uncompensated equilibrium in this economy is at the allocation W, where $(b_B, n_B) = (3, 1)$ and $(b_N, n_N) = (3, 3)$, with corresponding prices $(1, 3)$ for nuts and bolts respectively. This allocation is oligarchic — indeed, it is even "dictatorial" — because it is the best possible allocation for N given the need to respect the constraint that $b_B + n_B \geq 4$. Note that the initial endowment, however, is certainly not oligarchic because it is not even weakly Pareto efficient. Thus having the autarky allocation be non-oligarchic does not rule out the possibility that the only equilibrium may be oligarchic.

The allocation W is not only a Walrasian equilibrium at the price vector $(1, 3)$. It is also a compensated equilibrium with transfers at any non-negative price vector (p, q) with $0 \leq p \leq q$, including $(1, 1)$. For the particular compensated equilibrium price vector $(1, 1)$, of course, W is a cheapest point of B's feasible set. Since W is oligarchic, this is the compensated equilibrium price vector whose existence is assured by Prop. 4.4. Yet at the alternative compensated equilibrium price vector $(1, 3)$ even agent B has a cheaper point, and so this also gives an uncompensated equilibrium. Moreover, only this price vector is relevant to the existence of a Walrasian equilibrium, since it is only for the price vector

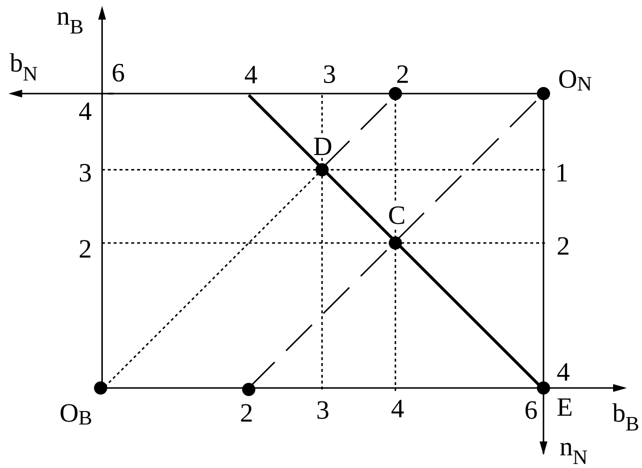

Figure 3

$(1,3)$ that the allocation W satisfies the budget constraint $p\, b_a + q\, n_a = p\, \bar{b}_a + q\, \bar{n}_a$ for each agent $a \in \{ B, N \}$ and for the appropriate endowment vector $(\bar{b}_a, \bar{n}_a)$.

This suggests that a weaker sufficient condition may still ensure existence of uncompensated equilibrium without transfers, since oligarchy by itself is not always an obstacle. Yet sometimes it certainly is, as illustrated by the alternative triangular Edgeworth box shown in Figure 3. The only difference from Figure 2 is that agent B's additional constraint has been changed from $b_B + n_B \geq 4$ to $b_B + n_B \geq 6$. This is enough, however, to prevent existence of uncompensated equilibrium without transfers. For the only compensated equilibrium without transfers occurs at C, where $(b_B, n_B) = (4, 2)$ and $(b_N, n_N) = (2, 2)$. This is indeed a cheapest point of B's feasible set at the only possible equilibrium price vector $(1, 1)$ which can be sustained without any transfers. It is not an uncompensated equilibrium, however, because at these prices agent B would like to move to the point D on the budget line DCE, where $(b_B, n_B) = (3, 3)$.

So we should look for a weaker sufficient condition which includes the oligarchic Walrasian allocation W in Figure 2, but excludes all allocations in the economy illustrated in Figure 3. Such a condition is the generalized form of irreducibility to be considered in the next Section.

5. Interdependent Agents and Irreducible Economies

5.1. Arrow and Debreu's Early Version of Irreducibility

It has already been pointed out that the first part of Arrow and Debreu (1954) relies on the implausible interiority assumption. But they do also present a different and more interesting second assumption. For this they assume that there is some set of goods $\mathcal{D}$ which all consumers desire. Also, that for any given feasible allocation, every consumer's endowment includes some kind of labour service which can be converted into an increased aggregate net supply of some good in $\mathcal{D}$, without any offsetting reduction in the aggregate net supply of any other good. In a finite economy, any such labour service is bound to command a positive wage provided that any consumer at all is maximizing preferences subject to his budget constraint.

Now, it was evident that with convex feasible sets and continuous preferences, for any consumer having a net trade vector of negative value — i.e, able to sell something of positive value — any compensated equilibrium must be an uncompensated equilibrium, as in the "cheaper point theorem" of Section 3.3.4 above, which actually requires only the weaker Assumption 1 of Section 3.3.2. So, in a finite economy, this second Arrow-Debreu assumption leaves just two possibilities. The first of these is that nobody has a cheaper point at the compensated equilibrium price vector, and that nobody is in uncompensated equilibrium. This is what happens in Arrow's exceptional case, but one could argue that the problem there is the attempt to price commodities such as return trips to the moon this year which cannot possibly be traded in any feasible allocation. The second possibility is more interesting. This is that every commodity can be traded, so every commodity can be supplied by somebody. Since at least one commodity should have a positive price, at least one consumer has a cheaper point, and so is in uncompensated equilibrium. But then, under the Arrow and Debreu assumption, all the desirable commodities in the set $\mathcal{D}$ must have positive prices, so everybody has a valuable labour service to supply, and therefore everybody has a cheaper feasible point. This shows that, in a finite economy, the compensated equilibrium is an uncompensated equilibrium for everybody, as required.

Arrow and Debreu's second assumption was unnamed. Yet it is actually a special case of the much more general irreducibility assumption introduced a few years later by McKenzie.

5.2. McKenzie's Irreducibility

As McKenzie (1959, p. 58) points out, he was "able to dispense with the categories of always desired goods and always productive goods used by Arrow and Debreu. An always desired good appears particularly implausible. It requires that every consumer be insatiable in this good within the supplies attainable by the whole market." One can also note that Arrow and Debreu's second assumption cannot apply in the usual kind of exchange economy in which each consumer has a fixed endowment vector of consumer goods, so that labour services are of no use to anyone.

In the introduction to McKenzie (1959, p. 55), he writes, "In loose terms, an economy is irreducible if it cannot be divided into two groups of consumers where one group is unable to supply any goods which the other group wants." After stating the relevant assumption formally, this verbal description is expanded somewhat on pp. 58–9:

> "[The irreducibility] assumption says that however we may partition the consumers into two groups if the first group receives an aggregate trade which is an attainable output for the rest of the market, the second group has within its feasible aggregate trades one which, if *added* to the goods already obtained by the first group, can be used to improve the position of someone in that group, while damaging the position of none there."

Later, in McKenzie's (1961) "Corrections," the formal assumption is re-stated in a way which "actually corresponds better with the verbal discussion," and certainly seems easier to interpret. Indeed, the economy is said to be *irreducible* if and only if, for every feasible allocation $\mathbf{x}$ and every partition $A_1 \cup A_2$ of A into two disjoint non-empty sets A_1 and A_2, there exists a pair $\mathbf{x}' \in \prod_{a \in A_1} X_a$ and $\mathbf{y} \in \prod_{a \in A_2} X_a$ with

$$\sum_{a \in A_1} x'_a + \sum_{a \in A_2} x_a = \sum_{a \in A_1} (x'_a - x_a) = -\sum_{a \in A_2} y_a$$

such that $x'_a \succsim_a x_a$ for all $a \in A_1$, and $x'_{a^*} \succ_{a^*} x_{a^*}$ for some $a^* \in A_1$. Equivalently, irreducibility requires that, for every feasible allocation $\mathbf{x}$ and every partition $A_1 \cup A_2$ of A, there exists $a^* \in A_1$ for whom

$$\sum_{a \in A_1} x_a = -\sum_{a \in A_2} x_a \in P_{a^*}(x_{a^*}) + \sum_{a \in A_1 \setminus \{a^*\}} U_a(x_a) + \sum_{a \in A_2} X_a,$$

or equivalently, for whom

$$0 \in P_{a^*}(x_{a^*}) + \sum_{a \in A_1 \setminus \{a^*\}} U_a(x_a) + \sum_{a \in A_2} [X_a + \{x_a\}].$$

Thus, the economy is irreducible if and only if, at every feasible allocation $\mathbf{x}$, every coalition which is a proper subset A_1 of A would have a Pareto improvement available for its own members provided it could suitably exploit *additional* resources from outside the economy which duplicate those which the complementary coalition A_2 could supply at the initial (autarky) allocation, while leaving the members of A_2 with their net trade vectors in the allocation $\mathbf{x}$. The force of this assumption is that every coalition A_2 has some initial resources with which the complementary coalition A_1 would be able to improve the existing allocation if it could have access to these additional resources from outside the economy, while leaving the coalition A_2 where it is. Provided some consumers are in uncompensated equilibrium, this will ensure that all the other consumers do have cheaper points.

Notice that $x_a \in U_a(x_a)$ for all $a \in A_2$. A weaker condition than irreducibility can be derived, therefore, by replacing the point set $\{x_a\}$ with $U_a(x_a)$ for all $a \in A_2$. The result is the requirement that, for every feasible allocation $\mathbf{x}$ and every partition $A_1 \cup A_2$ of A into two disjoint subsets, there should exist $a^* \in A_1$ for whom

$$\begin{aligned} 0 &\in P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in A_1 \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A_2} [X_a + U_a(x_a)] \\ &= P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in A \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A_2} X_a. \end{aligned}$$

It is this weakened condition which suggests the apparently more general idea of resource relatedness, to be considered next.

5.3. Interdependent Agents and Resource Relatedness

5.3.1. Interdependent Agents

Suppose that $\mathbf{x}$ is a feasible allocation. Let C be any proper subset of A. Then the coalition C is said to be *weakly independent* at $\mathbf{x}$ if and only if

$$0 \notin \sum\nolimits_{a \in C} P_a(x_a) + \sum\nolimits_{a \in A \setminus C} [U_a(x_a) + X_a].$$

And C is said to be *strongly independent* at $\mathbf{x}$ if and only if, for every $a^* \in C$, one has

$$0 \notin P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in A \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A \setminus C} X_a.$$

Thus a coalition C is independent at a given feasible allocation if there is no weakly Pareto superior allocation in which C benefits from access to additional resources from outside the economy which the complementary coalition $A \setminus C$ could have made available at the initial (autarkic) allocation.

Conversely, the set A of all agents is said to be *strongly* (resp. *weakly*) *interdependent* at the feasible allocation $\mathbf{x}$ if there is no proper subset C of A that is weakly (resp. strongly) independent at $\mathbf{x}$. Thus, agents are strongly interdependent at $\mathbf{x}$ if and only if, for every partition $A_1 \cup A_2$ of A into two disjoint non-empty sets A_1 and A_2,

$$0 \in \sum_{a \in A_1} P_a(x_a) + \sum_{a \in A_2} [U_a(x_a) + X_a].$$

And agents are weakly interdependent at $\mathbf{x}$ if and only if, for every partition $A_1 \cup A_2$ of A into two disjoint non-empty sets A_1 and A_2, there exists $a^* \in A_1$ for whom

$$0 \in P_{a^*}(x_{a^*}) + \sum_{a \in A \setminus \{a^*\}} U_a(x_a) + \sum_{a \in A_2} X_a.$$

In an allocation $\mathbf{x}$ at which all agents are interdependent, every coalition which is a proper subset of A has an improvement available among those allocations which would be weakly Pareto superior if the economy were able to exploit suitably *additional* resources which the complementary coalition could supply at the initial (autarky) allocation. The force of the interdependence assumption is that every coalition has some initial resources which the complementary coalition could use to reach an allocation that is weakly Pareto superior to the existing one, if only it could have access to these additional resources. Provided some consumers are in uncompensated equilibrium, this will ensure that the other consumers do have cheaper points.

5.3.2. Direct and Indirect Resource Relatedness

At any Pareto efficient allocation $\mathbf{x}$, the agent a^* is said to be *resource related* to $\bar{a}$ if there exists a pair $\mathbf{x}' \in \prod_{a \in A} X_a$ and $y_{\bar{a}} \in X_{\bar{a}}$, with $\sum_{a \in A} x'_a = -y_{\bar{a}}$, such that $x'_{a^*} \succ_{a^*} x_{a^*}$ and also $x'_a \succsim_a x_a$ for all $a \in A$.[10] And the agent a^* is said to be *indirectly resource related*

[10] This definition differs from Arrow and Hahn (1971, p. 117) in several ways, most of them unimportant. The main difference could be removed, however, provided that the requirement that $y_{\bar{a}} \in X_{\bar{a}}$ in the above definition were relaxed to the requirement that there exist corresponding finite sets of positive numbers λ^k and net trade vectors $x^k_{\bar{a}} \in X_{\bar{a}}$ ($k = 1$ to K) for which $y_{\bar{a}} = x_{\bar{a}} + \sum_{k=1}^{K} \lambda^k (x^k_{\bar{a}} - x_{\bar{a}})$. In other words, the main difference would be removed if the feasible set $X_{\bar{a}}$ were extended to the convex cone $K_{\bar{a}}(x_{\bar{a}})$ with vertex $x_{\bar{a}}$ which is generated by the feasible set $X_{\bar{a}}$. This difference would actually be irrelevant in an economy where every agent has convex preferences and a convex set of feasible net trades.

to $\bar{a}$ if there exists a chain $a_0, a_1, \ldots, a_n$ of agents in A, starting with $a_0 = a^*$ and ending with $a_n = \bar{a}$ such that, for $j = 1$ to n, agent a_{j-1} is resource related to a_j.

5.3.3. *Strong Interdependence Implies Resource Relatedness*

LEMMA. *If all agents are strongly interdependent at the feasible allocation* $\mathbf{x}$*, then all agents are resource related there.*

PROOF: If $\mathbf{x}$ is strongly interdependent, then for each $\bar{a} \in A$, the coalition $A \setminus \{\bar{a}\}$ cannot be weakly independent. This implies that there exists a pair $\mathbf{x}' \in \prod_{a \in A} X_a$ and $y_{\bar{a}} \in X_{\bar{a}}$, with $\sum_{a \in A} x'_a = -y_{\bar{a}}$, such that $x'_a \succsim_a x_a$ for all $a \in A$, and also $x'_a \succ_a x_a$ for all $a \in A \setminus \{\bar{a}\}$. This shows that every agent $a^* \in A \setminus \{\bar{a}\}$ is resource related to $\bar{a}$. Since this is true for all $\bar{a} \in A$, all agents are indeed resource related at the allocation $\mathbf{x}$. ∎

5.3.4. *Indirect Resource Relatedness Implies Weak Interdependence*

LEMMA. *If all agents are indirectly resource related at the feasible allocation* $\mathbf{x}$*, then all agents are weakly interdependent there.*

PROOF: Suppose that some proper subset C of A were strongly independent at the allocation $\mathbf{x}$. Let a^* be any member of C, and let $\bar{a}$ be any member of $A \setminus C$. Then there cannot be any pair $\mathbf{x}' \in \prod_{a \in A} X_a$ and $\mathbf{y} \in \prod_{a' \in A \setminus C} X_{a'}$ with $\sum_{a \in A} x'_a = -\sum_{a' \in A \setminus C} y_{a'}$ such that $x'_a \succsim_a x_a$ for all $a \in A$, and also $x'_{a^*} \succ_{a^*} x_{a^*}$. But $C \subset A$ and $\bar{a} \in A \setminus C$ together imply that $C \subset A \setminus \{\bar{a}\}$. Therefore there is no pair $\mathbf{x}' \in \prod_{a \in A} X_a$ and $y_{\bar{a}} \in X_{\bar{a}}$ with $\sum_{a \in A} x'_a = -y_{\bar{a}}$, such that $x'_a \succsim_a x_a$ for all $a \in A$, and also $x'_{a^*} \succ_{a^*} x_{a^*}$. This proves that, for all $a^* \in C$ and all $\bar{a} \in A \setminus C$, agent a^* is not resource related to $\bar{a}$. This contradicts indirect resource relatedness of all agents at the feasible allocation $\mathbf{x}$. ∎

5.3.5. *Why Weak Interdependence Does Not Imply Strong Interdependence*

Arrow and Hahn (1971, p. 119) provide a simple counter-example to illustrate how indirect resource relatedness does not imply (direct, or ordinary) resource relatedness. Since indirect resource relatedness implies weak interdependence, and strong interdependence implies resource relatedness, the same example shows that weak interdependence does not imply strong interdependence.

5.4. Generalized Interdependence

Having discussed earlier versions of irreducibility and resource relatedness, this section will now present the new generalized versions to be used in the rest of the paper. To show the need for such generalizations, note that not even resource relatedness is weak enough to be satisfied at the equilibrium point W in the clipped Edgeworth box economy illustrated in Figure 2 of Section 4.5. Additional resources duplicating those that can be supplied out of agent B's endowments include only bolts, and never any nuts. Yet more bolts are not enough for N, who also needs more nuts in order to become better off. The remedy is to allow N's *net trade vector* to be replicated as well. Since N can indeed be made better off when both agent B's endowment of bolts and N's compensated equilibrium net trade vector are replicated together, the sum of the two vectors must have positive value. Yet N's net trade vector must have zero value in compensated equilibrium, and so B's endowment must have positive value at the equilibrium price vector. This is precisely what needs to be proved.

First recall that, at the feasible allocation $\mathbf{x}$, agents are weakly interdependent if and only if, for every partition $A_1 \cup A_2$ of A into two disjoint non-empty sets A_1 and A_2, there exists $a^* \in A_1$ for whom

$$0 \in P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in A \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A_2} X_a.$$

For generalized interdependence, every set will be replaced by its convex hull, as in the definition of convexified oligarchy.[11] But an extra term $-\sum_{a \in A} \{\, 0, x_a \,\}$ will be added, to allow agents in the set A_1 to exploit replicas of their own or others' net trade vectors, if that would be desirable. Of course, $\{\, 0, x_a \,\}$ just denotes the pair set whose two members are 0 and x_a. In equilibrium without transfers, the value of such extra resources must be zero. Later in Prop. 5.5, the extra term will ensure that there are compensated equilibrium prices at which each agent's net trade vector has zero value.

[11] The idea that only directions matter is embodied, to some extent, in Arrow and Hahn's definition of resource relatedness, and in versions of irreducibility used in Bergstrom (1976) and Coles and Hammond (1986). It seems only natural to adapt this idea by considering convex hulls, which indeed is suggested by the definitions in Moore (1975) and McKenzie (1981, 1987). Since the only issue is whether 0 does or does not belong to a convex hull, any further extension to convex cones would make no difference.

Thus all agents are said to be *generalized interdependent* at the feasible allocation $\mathbf{x}$ if and only if, for every partition $A_1 \cup A_2$ of A into two disjoint non-empty sets A_1 and A_2, there exists $a^* \in A_1$ for whom

$$0 \in \mathrm{co}\left[P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in A \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A_2} X_a - \sum\nolimits_{a \in A} \{\, 0, x_a \,\}\right].$$

This definition leads to the following result, proved in Section 7.1:

PROPOSITION. *Under Assumptions 1 and 2, if* $(\mathbf{x}, p)$ *is a compensated equilibrium without transfers such that all agents are generalized interdependent at the allocation* $\mathbf{x}$*, then* $(\mathbf{x}, p)$ *must also be an uncompensated equilibrium.*

Thus, in an economy satisfying both Assumptions 1 and 2, generalized interdependence of all agents is a sufficient condition for a particular compensated equilibrium to be an uncompensated equilibrium. So generalized interdependence at all feasible allocations is sufficient for every compensated equilibrium to be an uncompensated equilibrium. In this case it follows, of course, that existence of a compensated equilibrium also implies existence of an uncompensated equilibrium.

5.5. Generalized Irreducibility

When interdependence was introduced in Section 5.3, it was as a weakening of irreducibility. Yet it will now be shown how a generalized version of the stronger condition of irreducibility is necessary to avoid some consumers having cheapest points in compensated equilibrium. Since generalized irreducibility will imply generalized interdependence trivially, this means that the two conditions are actually equivalent.

Recall from Section 5.2 how irreducibility requires that, for every feasible allocation $\mathbf{x}$ and every partition $A_1 \cup A_2$ of A, there exists $a^* \in A_1$ for whom

$$0 \in P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in A_1 \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A_2} [X_a + \{x_a\}].$$

Corresponding to the definition of generalized interdependence in the previous Section, say that the feasible allocation $\mathbf{x}$ is *generalized irreducible* if, for every partition $A_1 \cup A_2$ of A into disjoint non-empty sets, there exists $a^* \in A_1$ for whom

$$0 \in \mathrm{co}\Big[P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in A_1 \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A_2} (X_a + \{x_a\}) - \sum\nolimits_{a \in A} \{\, 0, x_a \,\}\Big].$$

This is indeed like generalized interdependence, except that the set $U_a(x_a)$ has been restricted to $\{x_a\}$ for all $a \in A_2$.

The following result, also proved in Section 7.2, is similar to Prop. 4.4. It shows how generalized irreducibility is a necessary condition for every agent to have a cheaper point at every compensated equilibrium price vector for which the value of each agent's net trade is zero.

PROPOSITION. *Let $\hat{\mathbf{x}}$ be a feasible allocation which is not generalized irreducible. Then there exists a coalition C and a price vector $p \neq 0$ such that $(\hat{\mathbf{x}}, p)$ is a compensated equilibrium without transfers in which all agents outside C are at cheapest points — i.e., for all $a \in A \setminus C$, it must be true that $x_a \in X_a \Longrightarrow p\, x_a \geq p\, \hat{x}_a = 0$.*

Since generalized interdependence is a sufficient, and generalized irreducibility is a necessary condition for a compensated equilibrium without transfers to be an uncompensated equilibrium, we are in the happy position of having a necessary condition that is stonger than the sufficient condition. Hence the two conditions must actually be both necessary and sufficient.

5.6. Sufficient Conditions for Survival

Recall how in Section 2.1.3 it was not assumed that each agent a can survive at every point of the set X_a of feasible net trades. Instead there will generally be a *survival set* $S_a \subset X_a$. Also, on the reasonable presumption that each agent prefers to survive, each preference ordering will satisfy the condition that $x'_a \succ_a x_a$ whenever $x'_a \in S_a$ but $x_a \in X_a \setminus S_a$. To ensure universal survival in equilibrium then requires additional assumptions, similar to those in McKenzie (1981, 1987) and Coles and Hammond (1986).[12] The first of these is the following obvious modification of the aggregate interiority Assumption 2.

ASSUMPTION 2S. *0 belongs to the interior of the convex hull of the set $S_A := \sum_{a \in A} S_a$.*

In addition, a rather different version of generalized interdependence is required. Indeed, at the particular feasible allocation $\mathbf{x}$, all agents are said to be *generalized interdependent with survival* at the feasible allocation $\mathbf{x}$ if and only if, for every partition $A_1 \cup A_2$

[12] McKenzie (1981, 1987) actually treats the feasible set X_a as if it were the survival set S_a, but then dispenses with the assumption that $0 \in X_a$. Our approach is effectively equivalent.

of A into two disjoint non-empty sets A_1 and A_2, there exists $a^* \in A_1$ for whom

$$0 \in \mathrm{co}\left[P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in A \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A_1} \{\, 0, x_a \,\} + \sum\nolimits_{a \in A_2} S_a\right].$$

The difference from the earlier definition of generalized interdependence comes in the requirement that, for each agent $a \in A_2$, the feasible set X_a has been replaced by the corresponding survival set S_a.

This definition leads to the following strengthening of Prop. 5.4 that is also proved in Section 7.1:

Proposition. *Under Assumptions 1 and 2S, if $(\mathbf{x}, p)$ is a compensated equilibrium without transfers such that all agents are generalized interdependent with survival at the allocation $\mathbf{x}$, then $(\mathbf{x}, p)$ must also be an uncompensated equilibrium in which $x_a \in S_a$ for all $a \in A$ — i.e., all agents survive.*

6. Continuum Economies

6.1. Preliminaries

6.1.1. A Measure Space of Consumers

As in Aumann (1964, 1966) and Hildenbrand (1974), it will now be assumed that there is a non-atomic measure space $(A, \mathcal{A}, \alpha)$ of economic agents in which A is the interval $[0, 1]$ of the real line $\Re$, $\mathcal{A}$ is the σ-algebra of Borel sets, and α is the Lebesgue measure satisfying $\alpha(A) = 1$.

6.1.2. Consumers' Feasible Sets

In addition to the earlier assumption that every consumer $a \in A$ has a fixed feasible set $X_a \subset \Re^G$ of net trade vectors x_a satisfying $0 \in X_a$, assume also that the graph $\{\, (a, x_a) \in A \times \Re^G \mid x_a \in X_a \,\}$ of the feasible set correspondence $X : A \twoheadrightarrow \Re^G$ is measurable when the space $A \times \Re^G$ is equipped with its product σ-algebra.

6.1.3. Consumers' Preferences

In addition to the earlier assumption that every consumer $a \in A$ has a weak preference ordering $\succsim_a$ satisfying local non-satiation on the set X_a, assume also that the preference relation $\succsim_a$ has a graph $\{ (a, x_a, x'_a) \in A \times X_a \times X_a \mid x'_a \succsim_a x_a \}$ which is a measurable subset of the space $A \times \Re^G \times \Re^G$ equipped with its product σ-algebra.

6.1.4. Feasible Allocations

A *feasible allocation* $\mathbf{x} : A \to \Re^G$ is a measurable function whose values satisfy $x_a \in X_a$ a.e. in A, and which also has the property that the mean net trade vector $\int_A x_a \alpha(da) = 0$. Note once again that only an exchange economy with individual production is being considered, and that free disposal is only possible to the extent that some individuals can dispose of goods freely.

6.1.5. Compensated and Uncompensated Equilibrium

In a continuum economy, an *uncompensated* (resp. *compensated*) *equilibrium* is a pair $(\mathbf{x}, p)$ consisting of a feasible allocation $\mathbf{x}$ and a price vector p such that $p\, x_a = 0$ and $x_a \in \xi^U_a(p, 0)$ (resp. $\xi^C_a(p, 0)$) a.e. in A.

An *uncompensated* (resp. *compensated*) *equilibrium with transfers* is a pair $(\mathbf{x}, p)$ consisting of a feasible allocation $\mathbf{x}$ and a price vector p such that, a.e. in A, one has $x_a \in \xi^U_a(p, p\, x_a)$ (resp. $\xi^C_a(p, p\, x_a)$).

6.1.6. The Aggregate Interiority Assumption

The counterpart for continuum economies of our earlier Assumption 2 in Section 3.4.1 is:

ASSUMPTION 2*. *0 belongs to the interior of the set $\int_A X_a\, \alpha(da)$.*

As in Section 3.4.2 this implies that, for any feasible allocation $\mathbf{x} : A \to \Re^G$ and at any price vector $p \neq 0$, the set

$$\{ a \in A \mid \exists \bar{x}_a \in X_a \text{ such that } p\, \bar{x}_a < p\, x_a \}$$

of individuals having cheaper feasible net trade vectors at the price vector p must have positive measure.

6.2. *Non-Oligarchic Allocations*

6.2.1. *Definitions*

A *coalition* in the continuum economy is a measurable set $C \in \mathcal{A}$ satisfying $0 < \alpha(C) < 1$. Such a coalition C is said to be an *oligarchy* at the feasible allocation $\mathbf{x}$ if, for every subset $A^* \subset C$ of positive measure, one has

$$0 \notin \int_{A^*} P_a(x_a)\,\alpha(da) + \int_{C\setminus A^*} U_a(x_a)\,\alpha(da) + \int_{A\setminus C} X_a\,\alpha(da).$$

The feasible allocation $\mathbf{x}$ in the continuum economy is said to be *non-oligarchic* if there is no oligarchy at $\mathbf{x}$. Note that there is no need to consider convexified oligarchies or convexified non-oligarchic allocations in a continuum economy.

6.2.2. *Sufficiency of Non-Oligarchy*

The following result, showing how non-oligarchy ensures that a compensated equilibrium is an uncompensated equilibrium, is proved in Section 7.2.

Proposition. *Under Assumptions 1 and 2*, if $(\mathbf{x}, p)$ is acompensated equilibrium with transfers in a continuum economy, and if the allocation $\mathbf{x}$ is non-oligarchic, then $(\mathbf{x}, p)$ must be an uncompensated equilibrium.*

6.2.3. *Necessity of Non-Oligarchy*

The following result, proved in Section 7.2, is similar to Prop. 4.4.

Proposition. *Let $\hat{\mathbf{x}}$ be a feasible allocation at which the coalition C is an oligarchy. Then there exists a price vector $p \neq 0$ such that $(\hat{\mathbf{x}}, p)$ is a compensated equilibrium with transfers in which almost all agents outside C are at cheapest points — i.e., for almost all $a \in A \setminus C$, it must be true that $x_a \in X_a \Longrightarrow p\,x_a \geq p\,\hat{x}_a$.*

6.3. Interdependence and Irreducibility

6.3.1. Irreducibility

The continuum economy is said to be *irreducible* (cf. Hildenbrand, 1972, p. 85) if, at every feasible allocation $\mathbf{x} : A \to \Re^G$ and for every partition $A_1 \cup A_2$ of A into two disjoint measurable sets A_1 and A_2 of positive measure, there exist two measurable functions $\mathbf{x}' : A_1 \to \Re^G$ and $\mathbf{y} : A_2 \to \Re^G$ such that:

(i) $\int_{A_1} x'_a \, \alpha(da) + \int_{A_2} x_a \, \alpha(da) = \int_{A_1} (x'_a - x_a) \, \alpha(da) = -\int_{A_2} y_a \, \alpha(da)$;

(ii) $x'_a \in X_a$ and $x'_a \succsim_a x_a$ a.e. in A_1, with $x'_a \succ_a x_a$ for almost all a in some subset A^* of A_1 which has positive measure;

(iii) $y_a \in X_a$ a.e. in A_2.

Equivalently, irreducibility requires that, at every feasible allocation $\mathbf{x} : A \to \Re^G$, and for every partition $A_1 \cup A_2$ of A into two disjoint measurable sets A_1 and A_2 of positive measure, there exists $A^* \subset A_1$ such that $\alpha(A^*) > 0$ and

$$\int_{A_1} x_a \, \alpha(da) = -\int_{A_2} x_a \, \alpha(da) \in \int_{A^*} P_a(x_a) \, \alpha(da) + \int_{A_1 \setminus A^*} U_a(x_a) \, \alpha(da) + \int_{A_2} X_a \, \alpha(da).$$

6.3.2. Generalized Interdependence and Irreducibility

The following definitions are obvious adaptations for continuum economies of those in Sections 5.3, 5.4 and 5.5 for finite economies.

All agents in the continuum economy are said to be *interdependent* at a particular feasible allocation $\mathbf{x}$ if, for every partition $A_1 \cup A_2$ of A into two disjoint sets of positive measure,

$$0 \in \int_{A^*} P_a(\hat{x}_a) \, \alpha(da) + \int_{A \setminus A^*} U_a(\hat{x}_a) \, \alpha(da) + \int_{A_2} X_a \, \alpha(da)$$

for some $A^* \subset A_1$ which has positive measure. Agents in the continuum economy are said to be *generalized interdependent* at a particular feasible allocation $\mathbf{x}$ if and only if, for every partition $A_1 \cup A_2$ of A into two disjoint sets of positive measure,

$$0 \in \int_{A^*} P_a(\hat{x}_a) \, \alpha(da) + \int_{A \setminus A^*} U_a(\hat{x}_a) \, \alpha(da) + \int_{A_2} X_a \, \alpha(da) - \int_{A} \{ 0, \hat{x}_a \} \, \alpha(da)$$

for some $A^* \subset A_1$ which has positive measure. Finally, the feasible allocation $\mathbf{x}$ in the continuum economy is said to be *generalized irreducible* if and only if, for every partition $A_1 \cup A_2$ of A into two disjoint sets of positive measure,

$$0 \in \int_{A^*} P_a(\hat{x}_a)\,\alpha(da) + \int_{A_1 \setminus A^*} U_a(\hat{x}_a)\,\alpha(da) + \int_{A_2} (X_a + \{\hat{x}_a\})\,\alpha(da) - \int_A \{\,0, \hat{x}_a\,\}\,\alpha(da)$$

for some $A^* \subset A_1$ which has positive measure.

6.3.3. Interdependent Agents and Uncompensated Equilibrium

For a continuum economy, the result corresponding to Prop. 5.4 is:

Proposition. *Under Assumptions 1 and 2*, if $(\mathbf{x}, p)$ is a compensated equilibrium without transfers such that all agents are generalized interdependent at the allocation $\mathbf{x}$, then $(\mathbf{x}, p)$ must also be an uncompensated equilibrium.*

6.3.4. Agents outside Independent Coalitions Are at Cheapest Points

The following last result is similar to Prop. 5.5.

Proposition. *Let $\hat{\mathbf{x}}$ be a feasible allocation in the continuum economy which is not generalized irreducible. Then there exists a coalition C and a price vector $p \neq 0$ such that $(\hat{\mathbf{x}}, p)$ is a compensated equilibrium without transfers in which almost all agents outside C are at cheapest points — i.e., for almost all $a \in A \setminus C$, it must be true that $x_a \in X_a \Longrightarrow p\,x_a \geq p\,\hat{x}_a = 0$.*

7. Proofs of Propositions

7.1. Sufficiency Theorems

PROP. 4.3. *Under Assumptions 1 and 2, any convexified non-oligarchic allocation* $\mathbf{x}$ *which is a compensated equilibrium with transfers at some price vector* $p \neq 0$ *must be an uncompensated equilibrium with transfers at the same price vector* p.

PROOF: (1) For the compensated equilibrium $(\mathbf{x}, p)$, let

$$C := \{ a \in A \mid \exists x'_a \in X_a : p\, x'_a < p\, x_a \}$$

be the set of agents $a \in A$ with cheaper points $x'_a \in X_a$. Assumption 2 and Lemma 3.4.2 together imply that C is not empty. Assumption 1, together with the cheaper point Prop. 3.3.4, implies that $(\mathbf{x}, p)$ is an uncompensated equilibrium for the members of C.

(2) Suppose that $A \setminus C$ is not empty. Since C cannot be a convexified oligarchy, there must exist an $a^* \in C$ for which

$$0 \in \mathrm{co} \left[P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in C \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A \setminus C} X_a \right].$$

This implies the existence of net trade vectors $x_a^j \in X_a$ (for $a \in A$ and $j = 1$ to J), and of positive scalars λ^j ($j = 1$ to J) with $\sum_{j=1}^J \lambda^j = 1$, such that:

(i) $0 = \sum_{j=1}^J \lambda^j \sum_{a \in A} x_a^j$;
(ii) $x_{a^*}^j \succ_{a^*} x_{a^*}$ for $j = 1$ to J;
(iii) $x_a^j \succsim_a x_a$ for all $a \in C \setminus \{a^*\}$ and for $j = 1$ to J.

(3) Because $(\mathbf{x}, p)$ is a compensated equilibrium, (iii) implies that $p\, x_a^j \geq p\, x_a$ for all $a \in C \setminus \{a^*\}$ and for $j = 1$ to J. Because $(\mathbf{x}, p)$ is an uncompensated equilibrium for the members of C, including agent a^*, (ii) implies that $p\, x_{a^*}^j > p\, x_{a^*}$ for $j = 1$ to J. Then, because $\sum_{a \in A} x_a = 0$, (i) implies that

$$\begin{aligned} &\sum\nolimits_{j=1}^J \lambda^j \sum\nolimits_{a \in A \setminus C} (p\, x_a^j - p\, x_a) \\ = - &\sum\nolimits_{j=1}^J \lambda^j \left[(p\, x_{a^*}^j - p\, x_{a^*}) + \sum\nolimits_{a \in C \setminus \{a^*\}} (p\, x_a^j - p\, x_a) \right] < 0. \end{aligned}$$

(4) Since each λ^j is positive, there must be at least one $a \in A \setminus C$ and at least one $x_a^j \in X_a$ for which $p\, x_a^j < p\, x_a$.

(5) This contradicts the assumption that no agent $a \in A \setminus C$ has any cheaper point in the feasible set X_a. Therefore $A \setminus C$ must be empty after all, and $(\mathbf{x}, p)$ must be an uncompensated equilibrium for all the members of A. ∎

Prop. 5.4. *Under Assumptions 1 and 2, if* $(\mathbf{x}, p)$ *is a compensated equilibrium without transfers such that all agents are generalized interdependent at the allocation* $\mathbf{x}$*, then* $(\mathbf{x}, p)$ *must also be an uncompensated equilibrium.*

Proof: (1) For the compensated equilibrium without transfers $(\mathbf{x}, p)$, let C be the set of agents $a \in A$ with cheaper points $x'_a \in X_a$ satisfying $p\,x'_a = 0$. As in (1) of the proof of Prop. 4.3, C must be non-empty and $(\mathbf{x}, p)$ must be an uncompensated equilibrium for the members of C.

(2) Suppose that $A \setminus C$ is not empty. Since C cannot be generalized independent, there must exist an $a^* \in C$ for whom

$$0 \in \mathrm{co}\Big[P_{a^*}(x_{a^*}) + \sum\nolimits_{a \in A \setminus \{a^*\}} U_a(x_a) + \sum\nolimits_{a \in A \setminus C} X_a - \sum\nolimits_{a \in A} \{\,0, x_a\,\}\Big].$$

This implies the existence of net trade vectors $x^j_a \in X_a$, $\bar{x}^j_a \in \{\,0, x_a\,\}$ (for $a \in A$) and $y^j_a \in X_a$ (for $a \in A \setminus C$), as well as of associated positive scalars λ^j ($j = 1$ to J) with $\sum_{j=1}^J \lambda^j = 1$, such that:

(i) $0 = \sum_{j=1}^J \lambda^j \Big[\sum_{a \in A} (x^j_a - \bar{x}^j_a) + \sum_{a \in A \setminus C} y^j_a\Big]$;

(ii) $x^j_{a^*} \succ_{a^*} x_{a^*}$ for $j = 1$ to J;

(iii) $x^j_a \succsim_a x_a$ for $a \in A \setminus \{a^*\}$ and for $j = 1$ to J;

(3) Because $(\mathbf{x}, p)$ is a compensated equilibrium without transfers, it must be true, as in part (3) of the proof of Prop. 4.3, that for $j = 1$ to J one has $p\,x^j_a \geq 0$ for all $a \in A \setminus \{a^*\}$ and also $p\,x^j_{a^*} > 0$. In addition, $p\,x_a = 0$ implies that $p\,\bar{x}^j_a = 0$ for all $a \in A$ and for $j = 1$ to J. Therefore, (i) implies that

$$\sum\nolimits_{j=1}^J \lambda^j \sum\nolimits_{a \in A \setminus C} p\,y^j_a = -\sum\nolimits_{j=1}^J \lambda^j \sum\nolimits_{a \in A} p\,(x^j_a - \bar{x}^j_a) < 0.$$

Thus there is at least one $a \in A \setminus C$ and at least one $y^j_a \in X_a$ for which $p\,y^j_a < 0$. The proof is then completed as in part (5) of the proof of Prop. 4.3 above. ∎

Prop. 5.6. *Under Assumptions 1 and 2S, if* $(\mathbf{x}, p)$ *is a compensated equilibrium without transfers such that all agents are generalized interdependent with survival at the allocation* $\mathbf{x}$*, then* $(\mathbf{x}, p)$ *must also be an uncompensated equilibrium in which* $x_a \in S_a$ *for all* $a \in A$ *— i.e., all agents survive.*

Proof: For the compensated equilibrium $(\mathbf{x}, p)$, let

$$C := \{\, a \in A \mid \exists x'_a \in S_a : p\,x'_a < p\,x_a = 0\,\}$$

be the set of agents $a \in A$ with cheaper points x'_a in their survival sets S_a.

The remainder of the proof is the same as that of Prop. 5.4, except that for each agent $a \in A \setminus C$ the feasible set X_a should be replaced by the corresponding survival set S_a. The conclusion is that $A \setminus C$ is empty, from which it follows that $(\mathbf{x}, p)$ is not only an uncompensated equilibrium, but is also one in which all the members of A can afford to survive and so will choose to do so. ▮

PROP. 6.2.2. *Under Assumptions 1 and 2*, if $(\mathbf{x}, p)$ is a compensated equilibrium with transfers in a continuum economy, and if the allocation $\mathbf{x}$ is non-oligarchic, then $(\mathbf{x}, p)$ must be an uncompensated equilibrium.*

PROP. 6.3.3. *Under Assumptions 1 and 2*, if $(\mathbf{x}, p)$ is a compensated equilibrium without transfers such that all agents are generalized interdependent at the allocation $\mathbf{x}$, then $(\mathbf{x}, p)$ must also be an uncompensated equilibrium.*

PROOF: The proofs of the two Props. 6.2.2 and 6.3.3 are exactly the same as those for Props. 4.3 and 5.4 respectively, except that: (i) $J = 1$ and so $\lambda^1 = 1$, since convex hulls need not be considered; (ii) the single agent $a^* \in C$ should be replaced by a set $A^* \subset C$ of positive measure; (iii) sums should be replaced by integrals in the obvious way; (iv) the phrases "for all" and "for at least one" in connection with any given set of agents should be replaced by "for almost all" and "for a set of positive measure" respectively; and (v) every reference to the set $A \setminus C$ being empty or non-empty should be replaced by the measure $\alpha(A \setminus C)$ being zero or positive. ▮

7.2. Necessity Theorems

PROP. 4.4. *Let $\hat{\mathbf{x}}$ be a feasible allocation at which the coalition C is a convexified oligarchy. Then there exists a price vector $p \neq 0$ such that $(\hat{\mathbf{x}}, p)$ is a weak compensated equilibrium with transfers in which all agents outside C are at cheapest points — i.e., for all $a \in A \setminus C$, it must be true that $x_a \in X_a \Longrightarrow p\, x_a \geq p\, \hat{x}_a$.*

PROOF: (1) For every $a^* \in C$, define the convex set

$$K_{a^*C}(\hat{\mathbf{x}}) := \mathrm{co}\left[P_{a^*}(\hat{x}_{a^*}) + \sum\nolimits_{a \in C \setminus \{a^*\}} U_a(\hat{x}_a) + \sum\nolimits_{a \in A \setminus C} X_a\right].$$

Because C is a convexified oligarchy at the feasible allocation $\hat{\mathbf{x}}$, it must be true that $0 \notin K_{a^*C}(\hat{\mathbf{x}})$ for every $a^* \in C$. Also $K_{a^*C}(\hat{\mathbf{x}})$ is non-empty because of non-satiation.

(2) Therefore there exists a price vector $p \neq 0$ and so a hyperplane $p\,y = 0$ through the point $0 \in \Re^G$ such that $p\,y \geq 0$ for all $y \in K_{a^* C}(\hat{\mathbf{x}})$. In particular, because $\sum_{a \in A} \hat{x}_a = 0$, it follows that $0 \leq \sum_{a \in A} p\,x_a = \sum_{a \in A} p\,(x_a - \hat{x}_a)$ whenever $x_{a^*} \in P_{a^*}(\hat{x}_{a^*})$, $x_a \in U_a(\hat{x}_a)$ (all $a \in C \setminus \{a^*\}$), and $x_a \in X_a$ (all $a \in A \setminus C$).

(3) Because $\hat{x}_{a^*}$ is on the boundary of $P_{a^*}(\hat{x}_{a^*})$, because $\hat{x}_a \in U_a(\hat{x}_a)$ for all $a \in C \setminus \{a^*\}$, and because also $\hat{x}_a \in X_a$ for all $a \in A \setminus C$, it then follows that: (i) $p\,x_{a^*} \geq p\,\hat{x}_{a^*}$ whenever $x_{a^*} \in P_{a^*}(\hat{x}_{a^*})$; (ii) for all $a \in C \setminus \{a^*\}$, one has $p\,x_a \geq p\,\hat{x}_a$ whenever $x_a \in U_a(\hat{x}_a)$; (iii) for all $a \in A \setminus C$, one has $p\,x_a \geq p\,\hat{x}_a$ whenever $x_a \in X_a$ and so also whenever $x_a \precsim_a \hat{x}_a$. Because of local non-satiation and Lemma 2.2.1, it follows that $(\hat{\mathbf{x}}, p)$ is indeed a compensated equilibrium with transfers in which all agents outside C are at cheapest points. ∎

PROP. 5.5. *Let $\hat{\mathbf{x}}$ be a feasible allocation which is not generalized irreducible. Then there exists a coalition C and a price vector $p \neq 0$ such that $(\hat{\mathbf{x}}, p)$ is a compensated equilibrium without transfers in which all agents outside C are at cheapest points — i.e., for all $a \in A \setminus C$, it must be true that $x_a \in X_a \Longrightarrow p\,x_a \geq p\,\hat{x}_a = 0$.*

PROOF: If $\hat{\mathbf{x}}$ is not generalized irreducible, then there must exist some coalition C such that $0 \notin K_{a^* C}(\hat{\mathbf{x}})$ for every $a^* \in C$, where $K_{a^* C}(\hat{\mathbf{x}})$ denotes the convex hull of the non-empty set

$$P_{a^*}(\hat{x}_{a^*}) + \sum_{a \in C \setminus \{a^*\}} U_a(\hat{x}_a) + \sum_{a \in A \setminus C} (X_a + \{\hat{x}_a\}) - \sum_{a \in A} \{\, 0, \hat{x}_a \,\}.$$

Arguing as in the proof of Prop. 4.4 above, there is a price vector $p \neq 0$ such that

$$\begin{aligned} 0 &\leq \sum_{a \in A} p\,x_a + \sum_{a \in A \setminus C} p\,\hat{x}_a - \sum_{a \in A} p\,y_a \\ &= \sum_{a \in A} p\,(x_a - \hat{x}_a) + \sum_{a \in A \setminus C} p\,\hat{x}_a - \sum_{a \in A} p\,(y_a - \hat{x}_a) \end{aligned}$$

whenever $x_{a^*} \in P_{a^*}(\hat{x}_{a^*})$, $x_a \in U_a(\hat{x}_a)$ (all $a \in C \setminus \{a^*\}$), $x_a \in X_a$ (all $a \in A \setminus C$), and $y_a \in \{\, 0, \hat{x}_a \,\}$ (all $a \in A$). Since one can choose $y_a = 0$ and $y_{a'} = \hat{x}_{a'}$ for all $a' \in A \setminus \{a\}$, this implies in particular that $p\,\hat{x}_a \geq 0$ for all $a \in A$. Because $\sum_{a \in A} \hat{x}_a = 0$, it follows that $\sum_{a \in A} p\,\hat{x}_a = 0$ and so $p\,\hat{x}_a = 0$ for all $a \in A$.

The rest of the proof is similar to that of Prop. 4.4. ∎

PROP. 6.2.3. *Let $\hat{\mathbf{x}}$ be a feasible allocation at which the coalition C is an oligarchy. Then there exists a price vector $p \neq 0$ such that $(\hat{\mathbf{x}}, p)$ is a compensated equilibrium with transfers in which almost all agents outside C are at cheapest points — i.e., for almost all $a \in A \setminus C$, it must be true that $x_a \in X_a \Longrightarrow p\,x_a \geq p\,\hat{x}_a$.*

PROOF: Because C is an oligarchy at the feasible allocation $\hat{\mathbf{x}}$, one has

$$0 \notin K_{A^*C}(\hat{\mathbf{x}}) := \int_{A^*} P^*_a(\hat{x}_a)\,\alpha(da) + \int_{C\setminus A^*} U_a(\hat{x}_a)\,\alpha(da) + \int_{A\setminus C} X_a\,\alpha(da)$$

for every measurable set $A^* \subset C$ such that $\alpha(A^*) > 0$. Note that, because α is a non-atomic measure, $K_{A^*C}(\hat{\mathbf{x}})$ is a convex set, and that it is non-empty because of non-satiation.

The rest of the proof is an obvious modification of that of Prop. 4.4. ∎

PROP. 6.3.4. *Let $\hat{\mathbf{x}}$ be a feasible allocation in the continuum economy which is not generalized irreducible. Then there exists a coalition C and a price vector $p \neq 0$ such that $(\hat{\mathbf{x}}, p)$ is a compensated equilibrium without transfers in which almost all agents outside C are at cheapest points — i.e., for almost all $a \in A \setminus C$, it must be true that $x_a \in X_a \Longrightarrow p\,x_a \geq p\,\hat{x}_a = 0$.*

PROOF: Because the feasible allocation $\hat{\mathbf{x}}$ is not generalized irreducible, there must be a coalition $C \subset A$ for which $0 < \alpha(C) < 1$ and also $0 \notin K_{A^*C}(\hat{\mathbf{x}})$ for every measurable $A^* \subset C$ such that $\alpha(A^*) > 0$, where $K_{A^*C}(\hat{\mathbf{x}})$ denotes the convex set

$$\int_{A^*} P_a(\hat{x}_a)\,\alpha(da) + \int_{C\setminus A^*} U_a(\hat{x}_a)\,\alpha(da) + \int_{A\setminus C} (X_a + \{\hat{x}_a\})\,\alpha(da) - \int_A \{\,0, \hat{x}_a\,\}\,\alpha(da).$$

The rest of the proof is then almost identical to that of Prop. 5.5 above. ∎

8. Concluding Remarks

McKenzie's assumption of irreducibility plays a crucial role in showing how compensated equilibria will be uncompensated equilibria because agents have cheaper net trade vectors in their feasible sets. This paper has considered several different versions of this assumption, including Arrow and Hahn's later notion of resource relatedness.

Actually, for considering a particular allocation which may be a compensated equilibrium with lump-sum transfers, the alternative simpler condition of non-oligarchy suffices, as was shown in Section 4. But the example of Section 4.5 illustrated how, in order to prove that every compensated equilibrium at which all agents are on their Walrasian budget constraint is an uncompensated equilibrium, it is enough to use more involved but weaker assumptions such as irreducibility and resource relatedness. This is because all individuals should have net trades whose value is zero at the equilibrium price voector. Survival of all individuals in compensated equilibrium without transfers can also be assured with a slight modification of these conditions.

Finally, these non-oligarchy and irreducibility assumptions were generalized to allow non-convexities in agents' feasible sets, and also to produce necessary conditions, in the absence of which some compensated equilibria will inevitably put some agents at cheapest points of their feasible sets. Of course, compensated equilibria could still be uncompensated equilibria even if some agents are at cheapest points. This is certainly true when, for instance, there is a unique cheapest point in each agent's feasible set, as there will be in the case (considered by Nikaido, 1956, 1957) where prices are strictly positive and each agent's consumption set is the non-negative orthant. But for the usual proofs of uncompensated equilibrium to be applicable, the important sufficient conditions of non-oligarchy and irreducibility are also necessary conditions. That is, there are no weaker assumptions which can be used to replace convexified non-oligarchy and generalized irreducibility in the standard proofs.

REFERENCES

K.J. Arrow (1951), "An Extension of the Basic Theorems of Classical Welfare Economics," in *Proceeedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* edited by J. Neyman (Berkeley: University of California Press), pp. 507–532; reprinted in Newman (1968) and in Arrow (1983).

K.J. Arrow (ed.) (1971), *Selected Readings in Economic Theory from* **Econometrica**. Cambridge, Mass.: M.I.T. Press.

K.J. Arrow (ed.) (1983), *Collected Papers of Kenneth J. Arrow, Vol. 2: General Equilibrium*. Cambridge, Mass.: Belknap Press of Harvard University Press.

K.J. Arrow and G. Debreu (1954), "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, **22**: 265–290; reprinted in Arrow (1983) and in Debreu (1983).

K.J. Arrow and F.H. Hahn (1971), *General Competitive Analysis*. San Francisco: Holden–Day.

R.J. Aumann (1964), "Markets with a Continuum of Traders," *Econometrica*, **32**: 39–50; reprinted in Newman (1968).

R.J. Aumann (1966), "Existence of Competitive Equilibria in Markets with a Continuum of Traders," *Econometrica*, **34**: 1–17.

T.C. Bergstrom (1976), "How to Discard 'Free Disposability' — At No Cost," *Journal of Mathematical Economics*, **3**: 131–134.

J. Broome (1972), "Existence of Equilibrium in Economies with Indivisible Commodities," *Journal of Economic Theory*, **5**: 224–250.

J.L. Coles and P.J. Hammond (1986), "Walrasian Equilibrium without Survival: Existence, Efficiency, and Remedial Policy," Stanford University, Institute of Mathematical Studies in the Social Sciences, Economics Technical Report No. 483.

G. Debreu (1951), "The Coefficient of Resource Utilization," *Econometrica*, **19**: 273–292; reprinted in Debreu (1983) and Arrow (1971).

G. Debreu (1954), "Valuation Equilibrium and Pareto Optimum," *Proceedings of the National Academy of Sciences*, **38**: 588–592; reprinted in Debreu (1983).

G. Debreu (1959), *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*. New York: John Wiley.

G. Debreu (1962), "New Concepts and Techniques for Equilibrium Analysis," *International Economic Review*, **3**: 257–273; reprinted in Debreu (1983).

G. Debreu (1983), *Mathematical Economics: Twenty Papers of Gerard Debreu*. Cambridge: Cambridge University Press.

D. Duffie (1988), *Security Markets: Stochastic Models*. San Diego: Academic Press.

D. Duffie and H. Sonnenschein (1989), "Arrow and General Equilibrium Theory," *Journal of Economic Literature*, **27**: 565–598.

B.C. Eaves (1976), "A Finite Algorithm for the Linear Exchange Model," *Journal of Mathematical Economics*, **3**: 197–203.

D. Gale (1955), "The Law of Supply and Demand," *Mathematica Scandinavica*, **3**: 155–169; reprinted in Newman (1968).

D. Gale (1957), "Price Equilibrium for Linear Models of Exchange," Technical Report P-1156, RAND Corporation.

D. Gale (1960), *The Theory of Linear Economic Models*. New York: McGraw-Hill.

D. Gale (1976), "The Linear Exchange Model," *Journal of Mathematical Economics*, **3**: 204–209.

F.R. Gantmacher (1953; English translation 1959), *The Theory of Matrices (2 vols.)*. New York: Chelsea.

W.M. Gorman (1953), "Community Preference Fields," *Econometrica*, **22**: 63–80.

F.D. Graham (1948), *The Theory of International Values*. Princeton: Princeton University Press.

J.-M. Grandmont and D. McFadden (1972), "A Technical Note on the Classical Gains from Trade," *Journal of International Economics*, **2**: 109–125.

P.J. Hammond, M. Kaneko and M.H. Wooders (1989), "Continuum Economies with Finite Coalitions: Core, Equilibrium, and Widespread Externalities," *Journal of Economic Theory*, **49**: 113–134.

C. Henry (1970), "Indivisibilités dans une économie d'échange," *Econometrica*, **38**: 542–558.

W. Hildenbrand (1972), "Metric Measure Spaces of Economic Agents," in *Proceeedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Vol. II* edited by L. Le Cam, J. Neyman, E. L. Scott (Berkeley: University of California Press), pp. 81–95.

W. Hildenbrand (1974), *Core and Equilibria of a Large Economy*. Princeton: Princeton University Press.

W. Hildenbrand and A.P. Kirman (1988), *Equilibrium Analysis: Variations on Themes by Edgeworth and Walras*. Amsterdam: North-Holland.

S. Honkapohja (1987), "On the Continuity of Compensated Demand," *International Economic Review*, **28**: 545–557.

S. Kakutani (1941), "A Generalization of Brouwer's Fixed Point Theorem," *Duke Mathematical Journal*, **8**: 457–459.

M.A. Khan (1989), "Lionel McKenzie on the Existence of Competitive Equilibrium," Johns Hopkins University, Department of Political Economy, preprint; presented to the Conference on "General Equilibrium and Growth: The Legacy of Lionel McKenzie" at the University of Rochester.

M.A. Khan and A. Yamazaki (1981), "On the Cores of Economies with Indivisible Commodities and a Continuum of Traders," *Journal of Economic Theory*, **24**: 218–225.

H.W. Kuhn and A.W. Tucker (1951), "Nonlinear Programming," in *Proceeedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* edited by J. Neyman (Berkeley: University of California Press), pp. 481–492, reprinted in Newman (1968).

L.W. McKenzie (1954a), "On Equilibrium in Graham's Model of World Trade and Other Competitive Systems," *Econometrica*, **22**: 147–161.

L.W. McKenzie (1954b), "Specialisation and Efficiency in World Production," *Review of Economic Studies*, **21**: 165–180.

L.W. McKenzie (1955), "Competitive Equilibrium with Dependent Consumer Preferences," in *Proceeedings of the Second Symposium on Linear Programming, Vol I* edited by H.A. Antosiewicz (Washington: National Bureau of Standards and Directorate of Management Analysis), pp. 277–294; reprinted in Newman (1968).

L.W. McKenzie (1957), "Demand Theory without a Utility Index," *Review of Economic Studies*, **24**: 185–189.

L.W. McKenzie (1959, 1961), "On the Existence of General Equilibrium for a Competitive Market," and "———: Some Corrections," *Econometrica*, **27**: 54–71 and **29**; 247–8; reprinted in Arrow (1971).

L.W. McKenzie (1981), "The Classical Theorem on Existence of Competitive Equilibrium," *Econometrica*, **49**: 819–841.

L.W. McKenzie (1987), "General Equilibrium," in *The New Palgrave: A Dictionary of Economics* edited by J. Eatwell, M. Milgate, and P. Newman (London: Macmillan).

A. Mas-Colell (1977), "Indivisible Commodities and General Equilibrium Theory," *Journal of Economic Theory*, **16**: 443–456.

J.C. Moore (1975), "The Existence of 'Compensated Equilibrium' and the Structure of the Pareto Efficient Frontier," *International Economic Review*, **16**: 267–300.

P. Newman (ed.) (1968), *Readings in Mathematical Economics, Vol. I: Value Theory*. Baltimore: Johns Hopkins Press.

H. Nikaido (1956, 1957), "On the Classical Multilateral Exchange Problem," and "A Supplementary Note to '———'," *Metroeconomica*, **8**: 21–45, and **9**: 209–210.

T. Rader (1964), "Edgeworth Exchange and General Economic Equilibrium," *Yale Economic Essays*, **4**: 133–180.

A. Spivak (1978), "A Note on Arrow-Hahn's Resource Relatedness (Or McKenzie's Irreducibility)," *International Economic Review*, **19**: 527–531.

A. Yamazaki (1978), "An Equilibrium Existence Theorem Without Convexity Assumptions," *Econometrica*, **46**: 541–555.

A. Yamazaki (1981), "Diversified Consumption Characteristics and Conditionally Dispersed Endowment Distribution: Regularizing Effect and Existence of Equilibria," *Econometrica*, **49**: 639–654.