Monte Carlo Simulation of Macroeconomic Risk with a Continuum of Agents: The Symmetric Case*

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Abstract Suppose a large economy with individual risk is modeled by a continuum of pairwise exchangeable random variables (i.i.d., in particular). Then the relevant stochastic process is jointly measurable only in degenerate cases. Yet in Monte Carlo simulation, the average of a large finite draw of the random variables converges almost surely. Several necessary and sufficient conditions for such "Monte Carlo convergence" are given. Also, conditioned on the associated Monte Carlo σ-algebra, which represents macroeconomic risk, individual agents' random shocks are independent. Furthermore, a converse to one version of the classical law of large numbers is proved.

Key words Large economy, continuum of agents, law of large numbers, exchangeability, joint measurability problem, de Finetti's theorem, Monte Carlo convergence, Monte Carlo σ-algebra.

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1 Introduction

Consider an economy with a continuum of agents. Suppose all agents face independent and identically distributed (i.i.d.) random shocks. In many economic applications one would like to invoke an exact version of the law of large numbers, and claim that the fraction who experience each possible shock should almost surely equal the probability of that shock.[1] However, as pointed out by Doob in [10] and [11], with further elaborations in [14], [19] and [29], the usual construction of a process with a continuum of i.i.d. random variables creates fundamental measurability difficulties. The first concerns joint measurability — namely, except in some trivial cases, such a process can never be jointly measurable with respect to the completion of the usual product σ-algebra on the joint space of parameters and samples.[2] The second problem concerns sample measurability — as shown in Theorem 2.2 in [10], the collection of samples whose corresponding sample functions are not Lebesgue measurable has outer measure one, so Lebesgue measure offers no basis for a meaningful concept of the mean or the distribution of a sample function. That is, the sample function giving each agent's individual shock may not be typically Lebesgue measurable, and thus the fraction of agents associated with each shock may not be well-defined.

Ad hoc examples can be found in the literature to show that there is no contradiction between the independence condition and the essential constancy of sample distributions — i.e., between the condition and conclusion in a natural statement of the exact law of large numbers.[3] However, the conclusion of the exact law of large numbers can also fail badly in different versions of such ad hoc examples.[4]

In [27]–[30], some rich product probability structures on the joint space of parameters and of samples are used to make independence compatible with joint measurability. Such enriched product probability spaces extend the usual product probability spaces, retain the common Fubini property of product probability measures, and also accommodate an abundance of nontrivial independent processes. Both the sample and joint measurability problems are automatically resolved by the required Fubini property. Also, the desired exact law of large numbers holds if and only if the random variables are independent almost surely.

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[1] See [14] and [19], as well as [1], pp. 2198–2199 and [28], pp. 502–503 for some well known references incorporating claims of this kind.

[2] See p. 57 in [11] and Proposition 1.1 in [29].

[3] See [1], [14], [16] and [19].

[4] [30] contains some detailed comments on such ad hoc examples, and on the difficulities of using a purely finitely additive measure-theoretic framework. In addition, the standard Birkhoff example shows that a continuum of mutually orthogonal random variables has Pettis integral equal to zero — see Example 5, p. 43 in [9] or Example 3.2.1, p. 33 in [31]. [20] discusses some of the economics literature concerned with the Pettis integral.

Consider a continuous parameter process with mutually independent random variables that is not taken from a framework where the usual Fubini property is already satisfied. As pointed out in Remark 3 of [17], it may not be possible to ensure that such a process is measurable by extending the usual product σ-algebra while retaining the usual "two-way" Fubini property. However, it is shown in [17] that a natural "one-way Fubini" property does guarantee a unique meaningful solution to the joint measurability problem, even for processes with random variables that are independent in a very weak sense.

The approach taken in this paper is inspired by the Monte Carlo method that is sometimes used to find numerical approximations to an ordinary multiple integral, especially when the integrand is of high dimension, or is complicated in some other way — see, for example, [15] for a recent survey. The basic Monte Carlo method computes the integral of a real-valued function by taking the average of the integrand evaluated at randomly selected points. Our purpose here is to extend this method in order to simulate macroeconomic uncertainty when many agents face individual risk which is modeled by a process with a continuum of random variables. Indeed, suppose that a sequence of the random variables is obtained by evaluating the process at randomly selected points of the parameter space. If the average of these random variables converges in the sense to be specified in Section 2 below, the process is said to be "Monte Carlo convergent".

Note that if a continuum of random variables forms an i.i.d. process, then any sequence taken from the continuum collection is obviously still i.i.d. When the common mean exists, the classical law of large numbers says that the arithmetic mean of the first n random variables in such a sequence converges to the common mean almost surely as n tends to infinity. This will trivially imply Monte Carlo convergence for the simple i.i.d. case.

The main purpose of this paper is to move beyond this obvious i.i.d. case and to consider Monte Carlo convergence for a large economy modeled by a continuum of (essentially) pairwise exchangeable random variables.[5] As shown by Proposition 2 in Section 5.3 below, such a process cannot be jointly measurable with respect to the usual product σ-algebra unless it is degenerate in the sense that almost all random variables are identical (and thus almost all agents' random shocks are perfectly correlated). However, based on some techniques developed in [17] to study a "one-way Fubini" extension of the relevant product probability space, several necessary and sufficient conditions for Monte Carlo convergence can be formulated and proved. We shall also define a corresponding Monte Carlo σ-algebra. This will be shown to simulate macroeconomic risk in the sense that individual agents' shocks, conditioned on this σ-algebra, are independent. It follows that the basic concept of stochastic independence will be characterized by triviality of the Monte Carlo limit — see Proposition 1 below.

[5] Systematic applications of exchangeability in economics are proposed in [24]. Other economic applications can be found in [6], [18] and [21].

In the rest of the paper, Section 2 describes the basic formulation and provides some essential definitions and assumptions. Then Section 3 describes some equivalence results in the basic i.i.d. case. Section 4 introduces and studies the basic properties of several concepts related to symmetry, pairwise exchangeability, and conditional independence. General results on the equivalence of these basic concepts with some special form of Monte Carlo convergence are presented in Section 5. Section 6 provides the proof of Theorem 1. Finally, a concluding assessment appears in Section 7.

2 Basic formulation

2.1 Two probability spaces

Let $(T, \mathcal{T}, \lambda)$ denote a probability space which is to be regarded as the parameter space for a process. In the case of an economy with a continuum of agents, it is usual to regard T as the set of agents, and to take $(T, \mathcal{T}, \lambda)$ as the Lebesgue unit interval — i.e., $T = [0, 1]$, while λ is Lebesgue measure applied to the complete σ-field $\mathcal{T}$ obtained from the usual Borel σ-field by adding all sets which are sandwiched between two Borel sets of equal Lebesgue measure. However, none of our results rely on $(T, \mathcal{T}, \lambda)$ being this Lebesgue unit interval — instead, it can be an entirely general non-atomic probability space. Indeed, in statistical mechanics it might be natural to label each particle by its position at any particular time, in which case T should be a compact subset of $\mathbb{R}^3$ with its appropriate σ-algebra, and with a probability measure which is the appropriately normalized product Lebesgue measure. In economies with a continuum of agents, or games with a continuum of players, especially if there is incomplete information, one could take T to be a metric space of agents' or players' possible types, including some feature or label that uniquely identifies each agent or player. Or T could be a hyperfinite Loeb space which is used to index the economic agents, as discussed in [1].

Let $(\Omega, \mathcal{A}, P)$ be the sample probability space. For example, it can be the product of a continuum of copies of some other basic probability space, or some extension of this product, or some other space entirely. As usual in probability theory, it is not necessary to specify in detail what the sample probability space is provided some general existence issues are resolved. We assume that the σ-algebra $\mathcal{A}$ is *complete* in the sense that, if $A \in \mathcal{A}$ with $P(A) = 0$, then $A' \in \mathcal{A}$ for all $A' \subset A$. Let $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ denote the usual product probability space. For simplicity, the completion of this product space is denoted in the same way.

2.2 A Polish space

The variables of interest in a continuum economy usually describe agents' allocations, or else their characteristics such as endowments, preferences or utility functions, or discount rates. In a game with a continuum of players, the variables of interest usually describe strategies, or else characteristics such as payoff functions. We assume that all such variables are members x of some general metric space (X, d).

We shall assume that this metric space is *complete* in the sense that *Cauchy sequences* satisfying $d(x_m, x_n) \to 0$ as $m, n \to \infty$ must converge to some point in X, and also *separable* in the sense that X is the closure of some countable subset. A topological space is called a *Polish* space if it is homeomorphic to a complete separable metric space. Let $\mathcal{B}$ denote the Borel σ-algebra of the Polish space (X, d). Then $\mathcal{B}$ can be generated by a countable collection of open sets in X. From now on, we usually ignore the metric d and refer to $(X, \mathcal{B})$ as a Polish space.

It should be remarked that many rich spaces are Polish. Examples include finite sets, finite-dimensional Euclidean spaces with their Euclidean metric, and the space of real-valued continous functions on a bounded interval with the usual supremum norm. One other important example is the space of real-valued functions that are right continuous and have left-hand limits — when this space is given its (metrizable) Skorohod topology — see [4], for example. Thus, our theory will encompass most of the standard models encountered in macroeconomics, with a continuum of agents all facing individual stochastic processes whose values lie in a suitable function space. Such stochastic processes can be Markov chains, Brownian motion, general Ito processes, Ito processes with random jumps, etc.

2.3 π- and λ-systems

Given any nonempty set Y, a π-*system* $\mathcal{P}$ is a family of subsets of Y that is closed under finite intersections. Let $\mathcal{D}$ be a non-empty family of subsets of Y, and let $\mathcal{D}^\pi$ denote the family of all finite intersections of sets in $\mathcal{D}$. Then, it is obvious that $\mathcal{D}^\pi$ is a π-*system*, and that $\mathcal{D}$ and $\mathcal{D}^\pi$ both generate the same σ-algebra $\sigma(\mathcal{D})$.

A family $\mathcal{Q}$ of subsets of Y is said to be a λ-*system* if it satisfies: (i) $Y \in \mathcal{Q}$; (ii) for $A, B \in \mathcal{Q}$ with $A \subseteq B$, $B \setminus A \in \mathcal{Q}$; (iii) for any sequence $\{A_n\}_{n=1}^\infty$ of pairwise disjoint sets in $\mathcal{Q}$, $\cup_{n=1}^\infty A_n \in \mathcal{Q}$.[6]

A result we use many times in this paper is:

Dynkin's π–λ Theorem: If $\mathcal{P}$ is a π-*system* and $\mathcal{Q}$ is a λ-*system* that contains $\mathcal{P}$, then $\mathcal{Q}$ must contain the σ-algebra $\sigma(\mathcal{P})$ generated by $\mathcal{P}$.[7]

This theorem allows one to infer that if any two finite signed measures coincide on $\mathcal{P} \cup \{Y\}$, then they must also coincide on $\sigma(\mathcal{P})$.

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[6] See, for example, [5], p. 41. This definition is easily seen to be equivalent to that of *Dynkin class*, in which (iii) is replaced by the requirement that for any sequence $\{A_n\}_{n=1}^\infty$ of sets in $\mathcal{Q}$ with $A_n \subseteq A_{n+1}$ for all n, $\cup_{n=1}^\infty A_n \in \mathcal{Q}$. See, for example, [8], p. 44.

[7] See, for example, [5], p. 42, or [8], p. 45.

2.4 Countably generated and essentially countably generated σ-algebras

Let $\mathcal{D}$ be a non-empty family of subsets of a space Y. If $\mathcal{D}$ is countable, then the σ-algebra $\sigma(\mathcal{D})$ generated by $\mathcal{D}$ is said to be *countably generated*.[8] Note that $\mathcal{D}^\pi$ is still countable and also generates $\sigma(\mathcal{D})$. Thus, we can always assume that a countably generated σ-algebra is generated by a countable π-system. In particular, if Y is a Polish space as in Section 2.2, then its Borel σ-algebra $\mathcal{Y}$ is countably generated from a countable π-system $\mathcal{O}$ of open sets in Y.

Consider any mapping $f : \Omega \to Y$. Let $\sigma(f)$ be the smallest σ-algebra $\mathcal{C}$ such that f is measurable w.r.t. $\mathcal{C}$ on Ω and $\mathcal{Y}$ on Y. This σ-algebra $\sigma(f)$ is called the σ-algebra *generated* by f. Since $\sigma(f)$ is also generated by the countable family $\{ f^{-1}(O) \mid O \in \mathcal{O} \}$, it is countably generated. On the other hand, if a σ-algebra $\mathcal{C}$ on Ω is countably generated, then there exists a Borel measurable mapping $\theta : \Omega \to [0, 1]$ such that $\mathcal{C} = \sigma(\theta)$ — see [5], Ex. 20.1, p. 270. In addition, if f is a random variable — i.e., f is measurable w.r.t. the σ-algebra $\mathcal{A}$ on Ω and the Borel σ-algebra $\mathcal{Y}$ on Y, then it is obvious that $\sigma(f)$ is a sub-σ-algebra of $\mathcal{A}$.

A sub-σ-algebra $\mathcal{C} \subseteq \mathcal{A}$ is said to be *essentially countably generated* if it is the *strong completion* of some countably generated σ-algebra $\mathcal{C}'$, in the sense that $\mathcal{C} = \{ A \in \mathcal{A} \mid \exists A' \in \mathcal{C}' : P(A \triangle A') = 0 \}$. For simplicity, from now on we describe a σ-algebra as countably generated even when it is only essentially countably generated. Of course, the extra sets in the essentially countably generated σ-algebra only differ from sets in the original countably generated σ-algebra by some null sets.

Let $\mathcal{M}(X, \mathcal{B})$ be the space of Borel probability measures on a Polish space $(X, \mathcal{B})$. We assume throughout that this space is equipped with the topology of weak convergence of measures. Indeed, $\mathcal{M}(X, \mathcal{B})$ with this topology is itself a Polish space — see, for example, [5], pp. 72–73. The following result on the measurability of mappings taking values in $\mathcal{M}(X, \mathcal{B})$ is often implicitly used in the literature. Since we are not able to find a precise reference, we give a proof here for the sake of completeness.

Lemma 1 *Let μ be a mapping from Ω to the space $\mathcal{M}(X, \mathcal{B})$ endowed with the topology of weak convergence of measures. Let $\mathcal{C}$ be the smallest σ-algebra on Ω such that for each $B \in \mathcal{B}$ the mapping $\omega \mapsto \mu_\omega(B)$ is $\mathcal{C}$-measurable from Ω to the real line with its Borel σ-algebra. Then $\mathcal{C} = \sigma(\mu)$, the σ-algebra on Ω generated by μ.*

Proof Let $\mathcal{F}$ be the family of closed sets in X. As shown in [4], p. 236, the specified topology of weak convergence on $\mathcal{M}(X, \mathcal{B})$ is generated by the family of subsets $\{\tau \in \mathcal{M}(X, \mathcal{B}) : \tau(F_i) < \rho(F_i) + \epsilon, i = 1, 2, \ldots, k\}$, where $\epsilon > 0$, $\rho \in \mathcal{M}(X, \mathcal{B})$, and the sets F_i are closed. It is also clear that $\omega \mapsto \mu_\omega$ is a measurable mapping into $\mathcal{M}(X, \mathcal{B})$ if and only if $\{\omega : \mu_\omega(F) < \rho(F) + \epsilon\}$ is measurable for each $\epsilon > 0$, each $\rho \in \mathcal{M}(X, \mathcal{B})$, and each closed set F in X.

[8] See the definition in [5], Ex. 2.11, p. 34.

This is equivalent to the condition that $\omega \mapsto \mu_\omega(F)$ is measurable for each closed set F in X. This implies that $\sigma(\mu)$ is equal to $\mathcal{C}_0$, defined as the smallest σ-algebra such that for each $F \in \mathcal{F}$ the mapping $\omega \mapsto \mu_\omega(F)$ is $\mathcal{C}_0$-measurable. From this definition, it is easy to verify that $\mathcal{C}_0 \subseteq \mathcal{C}$, and hence $\sigma(\mu) \subseteq \mathcal{C}$.

Consider the family $\mathcal{D} := \{B \in \mathcal{B} : \omega \mapsto \mu_\omega(B) \text{ is } \sigma(\mu)\text{-measurable}\}$. It is easy to verify that this family is a λ-system. We have just proved that $\mathcal{D}$ contains the family $\mathcal{F}$ of closed sets in X. Because $\mathcal{F}$ is a π-system, Dynkin's π–λ Theorem implies that $\sigma(\mathcal{F}) \subset \mathcal{D}$. But $\sigma(\mathcal{F}) = \mathcal{B}$ by definition of the Borel σ-algebra. So $\mathcal{D} = \mathcal{B}$, implying that $\omega \mapsto \mu_\omega(B)$ is $\sigma(\mu)$-measurable for each $B \in \mathcal{B}$. Hence $\mathcal{C} \subset \sigma(\mu)$.

Combining the results of these two paragraphs shows that $\mathcal{C} = \sigma(\mu)$, as required. □

2.5 Conditional expectations and regular conditional distributions

For the convenience of the reader, this subsection recapitulates the standard definitions of conditional expectation, conditional probability, and regular conditional distribution.[9]

Let $\mathcal{C}$ be a sub-σ-algebra of $\mathcal{A}$, and f an integrable real-valued function on $(\Omega, \mathcal{A}, P)$. An integrable real-valued function h on $(\Omega, \mathcal{A}, P)$ is said to be the *conditional expectation* of f given $\mathcal{C}$ if h is $\mathcal{C}$-measurable, and $\int_A f\,dP = \int_A h\,dP$ for all $A \in \mathcal{C}$. This h is essentially unique and usually denoted by $E(f|\mathcal{C})$.

For a π-system $\mathcal{C}^\pi$ that contains Ω and generates $\mathcal{C}$, if φ is a $\mathcal{C}$-measurable and P-integrable function on Ω that satisfies $\int_A f\,dP = \int_A \varphi\,dP$ for all $A \in \mathcal{C}^\pi$, then Dynkin's π–λ Theorem implies that $\varphi = E(f|\mathcal{C})$.

Let A be an event in $\mathcal{A}$. The *indicator function* $1_A : \Omega \to \{0, 1\}$ is defined by $1_A(\omega) = 1$ if $\omega \in A$, and $1_A(\omega) = 0$ otherwise. The *conditional probability* $P(A|\mathcal{C})$ of the event A given $\mathcal{C}$ is simply the conditional expectation $E(1_A|\mathcal{C})$ of this indicator function.

Let f be a random variable from Ω to a Polish space X with Borel σ-algebra $\mathcal{B}$. A mapping μ from Ω to $\mathcal{M}(X, \mathcal{B})$ is said to be a *regular conditional distribution* (r.c.d.) for f given $\mathcal{C}$ if for each fixed $B \in \mathcal{B}$, the mapping $\omega \mapsto \mu_\omega(B)$ is a version of $P(f^{-1}(B)|\mathcal{C})$ — i.e., $\mu_\omega(B) = E(1_{f^{-1}(B)}|\mathcal{C})$. A classical result of Doob says that an r.c.d. exists if the mapping takes values in a "nice" space, including any Polish space with its Borel σ-algebra — see [13], pp. 33 and 230. In particular, an r.c.d. exists for f given $\mathcal{C}$, which is denoted by $P(f^{-1}|\mathcal{C})$.

The following lemma is often used to compute conditional expectations — see [7], p. 223.

[9] The reader can find more of their properties set out in Chapter 7 of [7] and Chapter 4 of [13].

Lemma 2 *Suppose that f is a random variable from Ω to a Polish space X, and that $\mu = P(f^{-1}|\mathcal{C})$ is an r.c.d. for f given $\mathcal{C}$. Let φ be any real-valued function on X such that $\varphi(f)$ is integrable on $(\Omega, \mathcal{A}, P)$. Then $E(\varphi(f)|\mathcal{C}) = \int_X \varphi(x)\, d\mu_\omega(x)$.*

¿From Dynkin's π–λ Theorem, it is easy to obtain the following useful lemma.

Lemma 3 *Suppose that f is a random variable from Ω to a Polish space $(X, \mathcal{B})$. Suppose that $\mathcal{C}$ is a sub-σ-algebra of $\mathcal{A}$ which is generated by a π-system $\mathcal{C}^\pi$. Let $\mathcal{B}^\pi$ denote a countable π-system that generates $\mathcal{B}$. Let μ' be an r.c.d. for f given $\mathcal{C}$, and μ a mapping from Ω to $\mathcal{M}(X, \mathcal{B})$ such that $\mu_\omega(B)$ is $\mathcal{C}$-measurable for each $B \in \mathcal{B}^\pi$. Suppose finally that $P(C \cap f^{-1}(B)) = \int_C \mu_\omega(B) dP$ for each $C \in \mathcal{C}^\pi$ and $B \in \mathcal{B}^\pi$. Then μ is also an r.c.d. for f given $\mathcal{C}$, and $\mu_\omega = \mu'_\omega$ for P-a.e. $\omega \in \Omega$.*

Proof For each $C \in \mathcal{C}^\pi$ and $B \in \mathcal{B}^\pi$, since μ' is an r.c.d. for f given $\mathcal{C}$, one has

$$P(C \cap f^{-1}(B)) = \int_C 1_{f^{-1}(B)} dP = \int_C \mu'_\omega(B) dP,$$

Hence $\int_C \mu_\omega(B) dP = \int_C \mu'_\omega(B) dP$. Because $\mathcal{C}^\pi$ is a π-system, it follows that $\mu_\omega(B) = \mu'_\omega(B)$ for P-a.e. $\omega \in \Omega$. But $\mathcal{B}^\pi$ is a countable π-system, so we can group countably many P-null sets together to show that, for P-a.e. $\omega \in \Omega$, $\mu_\omega(B) = \mu'_\omega(B)$ holds simultaneously for all $B \in \mathcal{B}^\pi$, and hence for all $B \in \mathcal{B}$ by Dynkin's π–λ Theorem. □

2.6 A continuum of random variables

We assume throughout that the economic uncertainty of interest can be modeled as a process $g : T \times \Omega \to X$ with the property that, for each $t \in T$, the component mapping $\omega \mapsto g_t(\omega)$ is measurable, thus making every g_t a random variable defined on $(\Omega, \mathcal{A}, P)$. In this sense, provided that T has the cardinality of the continuum, we have a continuum of random variables g_t $(t \in T)$.[10]

2.7 Pairwise measurable probability

For simplicity, we also assume throughout that the process $g : T \times \Omega \to X$ has *pairwise measurable probabilities* in the sense that, for each $A \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$, the mapping $(t_1, t_2) \mapsto P(A \cap g_{t_1}^{-1}(B_1) \cap g_{t_2}^{-1}(B_2))$ is measurable w.r.t. the product σ-algebra $\mathcal{T} \otimes \mathcal{T}$ on the set of pairs $T \times T$.

To motivate this assumption, consider what happens when the pair of random variables g_{t_1} and g_{t_2} is sampled by drawing (t_1, t_2) at random from

[10] Actually, most of the results presented in this paper seem to require only that g_t is a random variable for almost all t.

the product space $(T, \mathcal{T}, \lambda)^2 = (T \times T, \mathcal{T} \otimes \mathcal{T}, \lambda \times \lambda)$, before ω is drawn at random from $(\Omega, \mathcal{A}, P)$. The pairwise measurable probability condition implies that, for each $A \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$, there should be a well-defined joint probability that $\omega \in A$ and that $g_{t_1}(\omega) \in B_1$, $g_{t_2}(\omega) \in B_2$. Of course, this joint probability is given by

$$\int_{T \times T} P(A \cap g_{t_1}^{-1}(B_1) \cap g_{t_2}^{-1}(B_2))\, d(\lambda \times \lambda)$$

It is important to realize that this measurability condition does *not* imply that, for P-a.e. fixed ω, the mapping $(t_1, t_2) \mapsto (g_{t_1}(\omega), g_{t_2}(\omega))$ is measurable with respect to the product σ-algebra $\mathcal{T} \otimes \mathcal{T}$ on $T \times T$, or even that almost all sample functions $t \mapsto g_\omega(t)$ are measurable with respect to the σ-algebra $\mathcal{T}$ on T.

2.8 Monte Carlo convergence and the Monte Carlo σ-algebra

We assume that the process g, and the associated continuum of random variables g_t $(t \in T)$, are intended to model an economy with many agents who face individual random shocks. It is natural to try to understand this process by considering what can be observed in a population formed by taking a random sequential draw from the agent space T. Such a general procedure may be called "Monte Carlo simulation" because of its similarity to the classical Monte Carlo method.

Take any set J in the product σ-algebra $\mathcal{T} \otimes \mathcal{B}$ on $T \times X$. Given a typical sequential draw $t^\infty \in T^\infty$, one can consider the finite sample $t_1, t_2, \ldots, t_n$ for each n. A relevant question then is whether the proportion of these n agents for whom the pair $(t, g(t, \omega))$ belongs to J will converge, as $n \to \infty$. This suggests the following:

Definition 1 *The process g is said to be* Monte Carlo convergent *if there is a mapping $\gamma : \Omega \to \mathcal{M}(T \times X, \mathcal{T} \otimes \mathcal{B})$ such that, for each $J \in \mathcal{T} \otimes \mathcal{B}$, the mapping $\omega \mapsto \gamma_\omega(J)$ is $\mathcal{A}$-measurable, and for λ^∞-a.e. sequence $t^\infty \in T^\infty$,*

$$\frac{1}{n} \sum_{i=1}^{n} 1_J(t_i, g(t_i, \omega)) \underset{P-a.s.}{\longrightarrow} \gamma_\omega(J).$$

In this case, we say that the measure-valued random variable γ_ω is the Monte Carlo limit *of g.*

For the processes considered in this paper, one only need consider the convergence in Definition 1 for those J in the form of rectangles $S \times B$ with $S \in \mathcal{T}, B \in \mathcal{B}$ (see Theorem 1 below). In this case, the convergence property simply holds for the fraction of the n randomly drawn random variables whose index t and value $x = g(t, \omega)$ lie in the sets S and in B respectively. This convergence property seems to be a minimal requirement for Monte Carlo simulation of the process g to make any sense at all.

When g is Monte Carlo convergent, it is natural to consider all the uncertainty as represented by the Monte Carlo limit.

Definition 2 *When the process g is Monte Carlo convergent, let $\mathcal{C}^g$ be the smallest σ-algebra such that the mapping $\omega \mapsto \gamma_\omega(J)$ is $\mathcal{C}^g$-measurable for each $J \in \mathcal{T} \otimes \mathcal{B}$. Then the sub-$\sigma$-algebra $\mathcal{C}^g$ of $\mathcal{A}$ is called the* Monte Carlo σ-algebra *generated by g.*

One important aim of this paper is to show that the Monte Carlo σ-algebra does simulate all the macroeconomic uncertainty in our setting.

3 The independent case

3.1 Monte Carlo simulation in the i.i.d. case

Let f be a mapping from $T \times \Omega$ to X with the property that, for each $t \in T$, the component mapping f_t is a random variable defined on $(\Omega, \mathcal{A}, P)$. Suppose that the random variables f_t $(t \in T)$ are i.i.d., with common distribution μ. That is, for each $n > 1$ and each $B_1, B_2, \ldots, B_n \in \mathcal{B}$, one has $P(\cap_{i=1}^n f_{t_i}^{-1}(B_i)) = \prod_{i=1}^n \mu(B_i)$ for any n points $t_1, t_2, \ldots, t_n \in T$.

Now let $\varphi : X \to \mathbb{R}$ be any μ-integrable function, with mean $m = \int_X \varphi(x) d\mu$. Then the functions defined by $h_t(\omega) := \varphi(f_t(\omega))$ (all $t \in T$) are also i.i.d. random variables, with common mean $m = \int_X \varphi(x) d\mu = \int_\Omega h_t(\omega) dP$.

Take any sequence $t^\infty = (t_1, t_2, \ldots,)$ from the countably infinite product space $(T, \mathcal{T}, \lambda)^\infty = (T^\infty, \mathcal{T}^\infty, \lambda^\infty)$ with $t_i \neq t_j$ for $i \neq j$. Then it is obvious that the sequence of random variables h_{t_i}, $i = 1, 2, \ldots$ is also i.i.d., with common mean m. Since λ is assumed to be non-atomic, the sequence h_{t_i} is i.i.d. for λ^∞-a.e. $t^\infty \in T^\infty$. When they are i.i.d., of course, the usual strong law of large numbers (see, for example, [13], Theorem 8.3 on p. 52) implies that the obvious sample average $\frac{1}{n} \sum_{i=1}^n h_{t_i}(\omega)$ converges P-a.s. to m as $n \to \infty$.

An important special case occurs when φ is the indicator function 1_B of some measurable set $B \in \mathcal{B}$. Then $\frac{1}{n} \sum_{i=1}^n 1_B(f_{t_i}(\omega))$ is the fraction of the n randomly drawn random variables whose value lies in B; this fraction must converge P-a.s. to the mean $\int_X 1_B(x) d\mu = \mu(B)$, which is the common probability that each $f_t(\omega)$ lies in B.

Furthermore, take any $S \in \mathcal{T}$. Then the usual strong law of large numbers implies that for λ^∞-a.e. $t^\infty \in T^\infty$, $\frac{1}{n} \sum_{i=1}^n 1_S(t_i)$ converges to $\lambda(S)$. Thus, for λ^∞-a.e. $t^\infty \in T^\infty$, the sequence $1_S(t_i)[1_B(f_{t_i}) - \mu(B)]$ $(i = 1, 2, \ldots)$ of uniformly bounded random variables is independent with mean zero. Another version of the law of large numbers (see [13], Theorem 8.2 on p. 52) therefore implies that the sequence $\frac{1}{n} \sum_{i=1}^n 1_S(t_i)[1_B(f_{t_i}(\omega)) - \mu(B)]$ converges P-a.s. to 0 as $n \to \infty$. Hence, for λ^∞-a.e. $t^\infty \in T^\infty$, $\frac{1}{n} \sum_{i=1}^n 1_S(t_i) 1_B(f_{t_i}(\omega))$ converges P-a.s. to $\lambda(S)\mu(B)$.[11] Of course, this result is just a version of the classical law of large numbers. Much more

[11] As noted in the equivalence of conditions (2) and (3) in Proposition 1, this means that the process g is Monte Carlo convergent, with constant limit $\lambda \times \mu$.

striking is the fact, shown below, that an "almost everywhere" or "essential" version of the i.i.d. condition is necessary for this convergence property to hold.

3.2 Necessary and sufficient conditions

The family of random variables g_t $(t \in T)$ is said to be *essentially pairwise independent* if the two random variables g_{t_1} and g_{t_2} are independent for $\lambda \times \lambda$-a.e. pair $(t_1, t_2) \in T \times T$.[12] If in addition there is a Borel probability measure μ on the Polish space $(X, \mathcal{B})$ such that g_t has distribution μ for λ-a.e. $t \in T$, then the process g is said to be *essentially i.i.d.*, and μ is the *essentially common distribution.*

Remark 1 *Let f be a mapping from $T \times \Omega$ to X with the property that each component function f_t is a random variable defined on $(\Omega, \mathcal{A}, P)$. When f is i.i.d., then f has pairwise measurable probabilities.*

Proof Let the mapping $(t_1, t_2, \omega) \mapsto F((t_1, t_2), \omega) := (f_{t_1}(\omega), f_{t_2}(\omega))$ define the process $F : T^2 \times \Omega \to X^2$ on the index space $(T, \mathcal{T}, \lambda)^2$ instead of on $(T, \mathcal{T}, \lambda)$. Because the random variables f_t $(t \in T)$ are mutually i.i.d., it follows that the random variables $F_{(t_1,t_2)}(\omega)$ $((t_1, t_2) \in T^2)$ must be essentially pairwise independent. We treat this F as the process g in [17]. Take $E = T^2 \times A \times (B_1 \times B_2)$, where $A \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$. Part (1) of Theorem 1 in [17] implies that the mapping on T^2 defined by $(t_1, t_2) \mapsto P(H^{-1}_{(t_1,t_2)}(E_{(t_1,t_2)}) = P(A \cap f^{-1}_{t_1}(B_1) \cap f^{-1}_{t_2}(B_2))$ is λ^2-integrable, so measurable w.r.t. $\mathcal{T} \otimes \mathcal{T}$. $\square$

The following result is an obvious implication of Theorems 1 and 2 in Section 5 below. It states that an essentially i.i.d. process is characterized by degeneracy of the Monte Carlo limit.

Proposition 1 *The following three conditions are equivalent:*

1. *the process g is essentially i.i.d., with an essentially common distribution μ;*
2. *for each $S \in \mathcal{T}$ and $B \in \mathcal{B}$, one has, for λ^∞-a.e. sequence $t^\infty \in T^\infty$,*

$$\frac{1}{n}\sum_{i=1}^{n} 1_S(t_i)\, 1_B(g(t_i, \omega)) \underset{P-a.s.}{\longrightarrow} \lambda(S)\mu(B).$$

3. *the process g is Monte Carlo convergent to the fixed product probability measure $\lambda \times \mu$ on $(T \times X, \mathcal{T} \otimes \mathcal{B})$.*

[12] A condition of this type is called "almost sure pairwise independence" in [27] and [28].

In an extended framework where the process g is jointly measurable and the usual Fubini property still holds, [27]–[30] show that essential pairwise independence is necessary as well as sufficient for an exact law of large numbers to hold. Proposition 1 is a counterpart of this result in the sequential or Monte Carlo setting considered in this paper. From another point of view, (1) $\Longrightarrow$ (2) is simply an obvious version of the "classical" law of large numbers restated in the continuum setting, while (2) $\Longrightarrow$ (1) is a converse of the classical law of large numbers in this setting.

4 Essential symmetry, pairwise exchangeability, and conditional independence

4.1 Essentially symmetric processes

The main focus of this paper is on general *essentially symmetric* processes $g : T \times \Omega \to X$ satisfying the condition that, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the probability $P(A \cap g_t^{-1}(B))$ is λ-a.e. independent of t. The following lemma characterizes such processes.

Lemma 4 *Let g be an essentially symmetric process. Then there exists a measurable mapping $\omega \mapsto \mu_\omega$ from $(\Omega, \mathcal{A})$ to $\mathcal{M}(X, \mathcal{B})$ such that, for each $A \in \mathcal{A}$, $B \in \mathcal{B}$, one has*

$$P(A \cap g_t^{-1}(B)) = \int_A \mu_\omega(B) dP$$

for λ-a.e. $t \in T$.

Proof For each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, define $c(A, B)$ as the common value of $P(A \cap g_t^{-1}(B))$, for λ-a.e. $t \in T$. For each fixed B, the mapping $A \mapsto c(A, B)$ is a measure on $(\Omega, \mathcal{A})$ which is absolutely continuous w.r.t. P. So there exists an essentially unique Radon–Nikodym derivative $\omega \mapsto \alpha^B(\omega)$ such that $c(A, B) = \int_A \alpha^B(\omega)\, dP$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Let $\mathcal{B}^\pi$ be a countable π-system that contains X and generates $\mathcal{B}$. Let $\mathcal{C}_0$ be the countably generated sub-σ-algebra of $\mathcal{A}$ generated by the family of mappings α^B ($B \in \mathcal{B}^\pi$). Let $\mathcal{C}_0^\pi$ be a countable π-system that contains Ω and generates $\mathcal{C}_0$. By grouping countably many λ-null sets together, we can find a set $T_0 \in \mathcal{T}$ with $\lambda(T_0) = 1$ such that, for each $t \in T_0$,

$$P(A \cap g_t^{-1}(B)) = \int_A \alpha^B(\omega)\, dP \tag{1}$$

for all $A \in \mathcal{C}_0^\pi, B \in \mathcal{B}^\pi$. Fix any $t_0 \in T_0$. Let μ be the regular conditional distribution $P(g_{t_0}^{-1} | \mathcal{C}_0)$ of g_{t_0} given $\mathcal{C}_0$. This means that μ is a measurable mapping from $(\Omega, \mathcal{C}_0)$ to $\mathcal{M}(X, \mathcal{B})$ such that for each $A \in \mathcal{C}_0$, $B \in \mathcal{B}$,

$$\int_A 1_{g_{t_0}^{-1}(B)} dP = P(A \cap g_{t_0}^{-1}(B)) = \int_A \mu_\omega(B) dP. \tag{2}$$

Hence by Equations (1) and (2), we have

$$\int_A \alpha^B(\omega)\, dP = \int_A \mu_\omega(B) dP \tag{3}$$

for all $A \in \mathcal{C}_0^\pi$, $B \in \mathcal{B}^\pi$.

Since $\mathcal{C}_0$ is generated by the π-system $\mathcal{C}_0^\pi$, Dynkin's π–λ Theorem implies that Equation (3) is still valid for all $A \in \mathcal{C}_0, B \in \mathcal{B}^\pi$. For each $B \in \mathcal{B}^\pi$, since both $\alpha^B(\omega)$ and $\mu_\omega(B)$ are $\mathcal{C}_0$-measurable, essential uniqueness of the Radon–Nikodym derivative implies that $\alpha^B(\omega) = \mu_\omega(B)$ for P-almost all $\omega \in \Omega$. This means that Equation (3) still holds for all $A \in \mathcal{A}$, $B \in \mathcal{B}^\pi$, so

$$P(A \cap g_t^{-1}(B)) = c(A, B) = \int_A \alpha^B(\omega)\, dP = \int_A \mu_\omega(B) dP \tag{4}$$

for λ-a.e. $t \in T$. Since $\mathcal{B}$ is generated by the π-system $\mathcal{B}^\pi$, Dynkin's π–λ Theorem implies that for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, equation (4) still holds for λ-a.e. $t \in T$. □

4.2 Essential pairwise exchangeability

A collection of random variables is said to be *pairwise exchangeable* if there is a common distribution π such that all pairs of random variables from the collection have the same joint distribution π. A natural extension of this definition is to say that the process g is *essentially pairwise exchangeable* if there exists a common joint probability measure π on $(X, \mathcal{B})^2$ such that almost all pairs of random variables in $\{g_t : t \in T\}$ have the same joint distribution π — i.e., for $\lambda \times \lambda$-a.e. $(t_1, t_2) \in T \times T$, one has $P(g_{t_1}^{-1}(B_1) \cap g_{t_2}^{-1}(B_2)) = \pi(B_1 \times B_2) = \pi(B_2 \times B_1)$ for all $B_1, B_2 \in \mathcal{B}$.

The following lemma shows that essential pairwise exchangeability implies essential symmetry.

Lemma 5 *If the process g is essentially pairwise exchangeable, then it is essentially symmetric.*

Proof Fix any $A \in \mathcal{A}$ and $B \in \mathcal{B}$. By definition of essential pairwise exchangeability, there exists a symmetric measure π on $(X, \mathcal{B})^2$ and a set T_1 with $\lambda(T_1) = 1$ such that, for each $t' \in T_1$, one has

$$P(g_{t'}^{-1}(B) \cap g_t^{-1}(B)) = E(1_{g_{t'}^{-1}(B)} 1_{g_t^{-1}(B)}) = \pi(B \times B) \tag{5}$$

for λ-a.e. $t \in T$, and also

$$P(g_{t'}^{-1}(B)) = E(1_{g_{t'}^{-1}(B)}) = \pi(B \times X) \tag{6}$$

Consider the Hilbert space $L_2(\Omega, \mathcal{A}, P)$, and let L be the smallest closed linear subspace which contains both the constant function $1 = 1_\Omega$ and also the family of indicator functions $\{ 1_{g_t^{-1}(B)} \mid t \in T_1 \}$. Let the function

$h : \Omega \to \mathbb{R}$ be the orthogonal projection of the indicator function 1_A onto L, with $h^\perp$ as its orthogonal complement. Then $1_A = h + h^\perp$ where, by definition, $h^\perp$ is orthogonal to each member of L. That is, $0 = E(h^\perp 1) = \int_\Omega h^\perp dP$ and also $0 = E(h^\perp 1_{g_t^{-1}(B)}) = \int_\Omega h^\perp 1_{g_t^{-1}}(B) dP$ for all $t \in T_1$. Because $1_A = h + h^\perp$, it follows that $E(1_A\, 1_{g_t^{-1}(B)}) = E(h\, 1_{g_t^{-1}(B)})$ for all $t \in T_1$, and also $P(A) = E 1_A = E(1_A\, 1) = E(h\, 1) = Eh$.

Next, because $h \in L$, there exists a sequence of functions

$$h_n = r_n\, 1 + \sum_{k=1}^{i_n} \alpha_n^k\, 1_{g_{t_n^k}^{-1}(B)} \ (n = 1, 2, \ldots)$$

with $t_n^k \in T_1$, as well as r_n and α_n^k $(k = 1, \ldots, i_n)$ all real constants, such that $h_n \to h$ in the norm of $L_2(\Omega, \mathcal{A}, P)$ — that is, $\int_\Omega (h_n - h)^2 dP \to 0$.

Let T_n^k be the set of t for which (5) holds when $t' = t_n^k$. By hypothesis, $\lambda(T_n^k) = 1$ because each $t_n^k \in T_1$. Define $T^* := T_1 \cap \left(\cap_{n=1}^\infty \cap_{k=1}^{i_n} T_n^k \right)$. Because T^* is the intersection of a countable family of sets all having measure 1 w.r.t. λ, it follows that $\lambda(T^*) = 1$. Also, for any $t \in T^*$, one has

$$P(A \cap g_t^{-1}(B)) = E(1_A 1_{g_t^{-1}(B)}) = E(h\, 1_{g_t^{-1}(B)}) = \lim_{n \to \infty} E(h_n 1_{g_t^{-1}(B)}) \quad (7)$$

But (5) and (6) both hold whenever $t \in T^*$ and $t' = t_n^k$, so

$$\begin{aligned} E(h_n 1_{g_t^{-1}(B)}) &= r_n\, E(1_{g_t^{-1}(B)}) + \sum_{k=1}^{i_n} \alpha_n^k\, E(1_{g_{t_n^k}^{-1}(B)}\, 1_{g_t^{-1}(B)}) \\ &= r_n\, \pi(B \times X) + \sum_{k=1}^{i_n} \alpha_n^k\, \pi(B \times B). \end{aligned}$$

It follows that $E(h_n 1_{g_t^{-1}(B)})$ is independent of t, for all $t \in T^*$. But then $P(A \cap g_t^{-1}(B))$ must have the same property, by (7). Since $\lambda(T^*) = 1$, this completes the proof. □

4.3 Essential conditional independence

For a given sub-σ-algebra $\mathcal{C} \subset \mathcal{A}$, two random variables $f_1, f_2 : \Omega \to X$ are said to be *conditionally independent* given $\mathcal{C}$ if, for every pair of Borel sets $B_1, B_2 \in \mathcal{B}$, the conditional probabilities satisfy

$$P(f_1^{-1}(B_1) \cap f_2^{-1}(B_2) | \mathcal{C}) = P(f_1^{-1}(B_1) | \mathcal{C})\, P(f_2^{-1}(B_2) | \mathcal{C}) \quad (8)$$

Lemma 6 *Suppose that $\mathcal{B}^\pi$ is a countable π-system that generates $\mathcal{B}$, and that $\mathcal{C}$ is a sub-σ-algebra of $\mathcal{A}$. For any two random variables $f_1, f_2 : \Omega \to X$, the following are equivalent:*

1. *the product r.c.d.* $P(f_1^{-1}|\mathcal{C}) \times P(f_2^{-1}|\mathcal{C})$ *is an r.c.d. for* (f_1, f_2) *given* $\mathcal{C}$*;*
2. f_1 *and* f_2 *are conditionally independent given* $\mathcal{C}$*;*
3. *equation (8) holds for all* $B_1, B_2 \in \mathcal{B}^\pi$*.*

Proof (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are obvious. (3) $\Longrightarrow$ (1) follows from Lemma 3. $\square$

The process g is said to be *essentially conditionally independent* given $\mathcal{C}$ if, for $(\lambda \times \lambda)$-a.e. $(t_1, t_2) \in T \times T$, the pair g_{t_1}, g_{t_2} is conditionally independent given $\mathcal{C}$. If, in addition, there is a mapping μ from Ω to $\mathcal{M}(X, \mathcal{B})$ such that μ is an r.c.d. for g_t given $\mathcal{C}$ for λ-a.e. $t \in T$, then g is said to be *essentially i.i.d. conditioned on* $\mathcal{C}$.

Lemma 7 *Suppose that there exists a countably generated* σ*-algebra* $\mathcal{C}'$ *such that the process* g *is essentially i.i.d. conditioned on* $\mathcal{C}'$*. Then* g *is essentially pairwise exchangeable.*

Proof By hypothesis, there is a $\mathcal{C}'$-measurable mapping α from Ω to $\mathcal{M}(X, \mathcal{B})$ which, for λ-a.e. $t \in T$, is a version of the regular conditional distribution of g_t given $\mathcal{C}'$. Since g is essentially i.i.d. conditioned on $\mathcal{C}'$, one has $P(\,(g_t, g_{t'})^{-1}|\mathcal{C}') = \alpha \times \alpha$ for $(\lambda \times \lambda)$-a.e. (t, t'). Thus, for each $V \in \mathcal{B} \otimes \mathcal{B}$,

$$P(\,(g_t, g_{t'})^{-1}(V)\,) = \int_\Omega (\alpha_\omega \times \alpha_\omega)(V)\, dP.$$

Hence $P(\,(g_t, g_{t'})^{-1}\,)$ is equal to the symmetric probability measure π defined by $\pi(V) := \int_\Omega (\alpha_\omega \times \alpha_\omega)(V)\, dP$ for each $V \in \mathcal{B} \otimes \mathcal{B}$. $\square$

5 Main Results

5.1 First equivalence theorem

The first equivalence theorem is stated for a given measurable mapping μ from $(\Omega, \mathcal{A})$ to the space $\mathcal{M}(X, \mathcal{B})$ of measures on the Polish space $(X, \mathcal{B})$ equipped with the Borel σ-algebra corresponding to the topology of weak convergence of measures. The proof of this theorem is through Lemmas 8, 10 and 11 in the next section.

Theorem 1 *Suppose* $\omega \mapsto \mu_\omega$ *is a measurable mapping from* $(\Omega, \mathcal{A})$ *to* $\mathcal{M}(X, \mathcal{B})$*. Let* $\mathcal{C}$ *be the* σ*-algebra which is (countably) generated by this mapping. Then the following conditions are equivalent:*

1. *for each* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$*, one has* $P(A \cap g_t^{-1}(B)) = \int_A \mu_\omega(B)\, dP$ *for* λ*-a.e.* $t \in T$*;*
2. *the process* g *is essentially i.i.d. conditioned on* $\mathcal{C}$*, with* $P(g_t^{-1}|\mathcal{C}) = \mu_\omega$ *for* λ*-a.e.* $t \in T$*;*
3. *for each* $S \in \mathcal{T}$*,* $B \in \mathcal{B}$*, and for* λ^∞*-a.e. sequence* $t^\infty \in T^\infty$*, one has*

$$\frac{1}{n} \sum_{i=1}^{n} 1_S(t_i)\, 1_B(g(t_i, \omega)) \underset{P-a.s.}{\longrightarrow} \lambda(S)\, \mu_\omega(B)$$

4. *the process* g *is Monte Carlo convergent, with Monte Carlo limit given by the product probability measure* $\lambda \times \mu_\omega$ *on* $(T \times X, \mathcal{T} \otimes \mathcal{B})$*.*

5.2 Second equivalence theorem

The second equivalence theorem uses the first to give necessary and sufficient conditions for the process to be essentially pairwise exchangeable. The equivalence of Conditions 1 and 4 below is a version of the classical De Finetti theorem which is appropriate in our setting, with a continuum of random variables.

Theorem 2 *The following four conditions are equivalent:*

1. *the process* g *is essentially pairwise exchangeable;*
2. *the process* g *is essentially symmetric;*
3. *there exists a measurable mapping* μ *from* $(\Omega, \mathcal{A})$ *to* $\mathcal{M}(X, \mathcal{B})$*, together with the corresponding countably generated* σ*-algebra* $\mathcal{C} = \sigma(\mu)$*, such that all four equivalent conditions of Theorem 1 are satisfied;*
4. *there exists a countably generated* σ*-algebra* $\mathcal{C}'$ *such that the process* g *is essentially i.i.d. conditioned on* $\mathcal{C}'$*.*

Proof (1) $\Longrightarrow$ (2) was shown in Lemma 5. By Lemma 4, (2) implies Condition 1 in Theorem 1, so (2) $\Longrightarrow$ (3). Condition 2 in Theorem 1 trivially implies (4), so (3) $\Longrightarrow$ (4). Finally, (4) $\Longrightarrow$ (1) was shown in Lemma 7. □

The following corollary follows easily from Theorems 1 and 2.

Corollary 1 *Assume that the process* g *is essentially pairwise exchangeable. Then*

1. *the Monte Carlo* σ*-algebra* $\mathcal{C}^g$ *equals* $\sigma(\mu)$*, where* μ *is the measurable mapping from* $(\Omega, \mathcal{A})$ *to* $\mathcal{M}(X, \mathcal{B})$*, as in the statement of Theorem 1;*
2. *the process* g *is essentially i.i.d. conditioned on* $\mathcal{C}^g$*.*

When a large economy with individual risk is modeled by an essentially pairwise exchangeable process g, this result shows that the corresponding Monte Carlo σ-algebra $\mathcal{C}^g$ does simulate macroeconomic uncertainty, in the sense that individual agents' random shocks are independent conditioned on $\mathcal{C}^g$.

5.3 Joint measurability implies perfect correlation

The following proposition shows that a process g satisfying the conditions stated above cannot be jointly measurable except in the completely trivial case when almost all the random variables g_t equal some fixed random variable, and so are perfectly correlated.

Proposition 2 *Suppose that the process g is $\mathcal{T} \otimes \mathcal{A}$-measurable, and satisfies any of the equivalent conditions of Theorem 2. Then there is a random variable α from Ω to X such that for λ-a.e. $t \in T$, $g_t(\omega) = \alpha(\omega)$ for P-a.e. $\omega \in \Omega$.*

Proof By Theorem 2, condition 1 of Theorem 1 must be satisfied. So there exists a measurable mapping $\omega \mapsto \mu_\omega$ from $(\Omega, \mathcal{A})$ to $\mathcal{M}(X, \mathcal{B})$ such that, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$P(A \cap g_t^{-1}(B)) = \int_A \mu_\omega(B) dP$$

for λ-a.e. $t \in T$. Take any $S \in \mathcal{T}$. Because g is assumed to be $\mathcal{T} \otimes \mathcal{A}$-measurable, one can use the Fubini theorem and integrate the above equation over S to obtain

$$\int_{S \times A} 1_{g^{-1}(B)} d(\lambda \times P) = \int_S P(A \cap g_t^{-1}(B)) d\lambda = \int_{S \times A} \mu_\omega(B) d(\lambda \times P).$$

Since all the measurable rectangles $S \times A$ form a π-system, Dynkin's π–λ Theorem implies that for each $B \in \mathcal{B}$ one has

$$\int_F 1_{g^{-1}(B)} \, d(\lambda \times P) = \int_F \mu_\omega(B) \, d(\lambda \times P)$$

for all $F \in \mathcal{T} \otimes \mathcal{A}$. So, by essential uniqueness of the Radon–Nikodym derivative, it follows that for each $B \in \mathcal{B}$ one has

$$1_{g^{-1}(B)}(t, \omega) = \mu_\omega(B) \tag{9}$$

for $\lambda \times P$-a.e. $(t, \omega) \in T \times \Omega$.

Let d be a metric on X and $\{x_n\}_{n=1}^\infty$ a dense sequence in X. Let $\mathcal{B}^\pi$ be the countable collection of all the open balls $B(x_n, 1/m)$ centered at x_n and with radius $1/m$, for $n, m \geq 1$. By grouping together countably many $(\lambda \times P)$-null sets, one can show that there exists a set $D \in \mathcal{T} \otimes \mathcal{A}$ with $(\lambda \times P)(D) = 1$ such that for each $(t, \omega) \in D$, equation (9) holds simultaneously for all $B \in \mathcal{B}^\pi$.

Consider any $(t, \omega) \in D$. Suppose that $t' \in T$ is such that $g_\omega(t') \neq g_\omega(t)$. Then there exists a ball $B_0 \in \mathcal{B}^\pi$ such that $g_\omega(t) \in B_0$ but $g_\omega(t') \notin B_0$. Our hypotheses imply that

$$\mu_\omega(B_0) = 1_{g_\omega^{-1}(B_0)}(t) = 1 \neq 1_{g_\omega^{-1}(B_0)}(t') = 0$$

and so $(t', \omega) \notin D$. On the other hand, therefore, if $(t, \omega), (t', \omega) \in D$, then $g_\omega(t) = g_\omega(t')$. Hence, for P-a.e. $\omega \in \Omega$ there is a point $\alpha(\omega) \in X$ such that $g(t, \omega) = \alpha(\omega)$ for all $(t, \omega) \in D$. Because $(\lambda \times P)(D) = 1$, the Fubini Theorem implies that $P(D_t) = 1$ for λ-a.e. $t \in T$. Hence, for λ-a.e. $t \in T$, one has $g_t(\omega) = \alpha(\omega)$ for P-a.e. $\omega \in \Omega$. Since g_t is $\mathcal{A}$-measurable for λ-a.e. $t \in T$, the function α must also be $\mathcal{A}$-measurable. □

6 Proof of Theorem 1

6.1 Proof that (1) $\Longrightarrow$ (2)

Lemma 8 *Suppose that for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, one has*

$$P(A \cap g_t^{-1}(B)) = \int_A \mu_\omega(B)\, dP. \tag{10}$$

for λ-a.e. $t \in T$. Then the process g is essentially i.i.d. conditioned on $\mathcal{C}$, with $P(g_t^{-1}|\mathcal{C}) = \mu_\omega$ for λ-a.e. $t \in T$.

Proof Let $\mathcal{C}^\pi = \{C_n\}_{n=1}^\infty$ and $\mathcal{B}^\pi = \{B_m\}_{m=1}^\infty$ be countable π-systems for $\mathcal{C}$ and $\mathcal{B}$ respectively. For each pair (m, n), there exists a set T_{mn} with $\lambda(T_{mn}) = 1$ such that for all $t \in T_{mn}$, equation (10) holds with $A = C_n$ and $B = B_m$. So for any $t \in T^* := \cap_{m=1}^\infty \cap_{n=1}^\infty T_{mn}$, equation (10) holds whenever $A = C_n$ and $B = B_m$, for all pairs (m, n) simultaneously. Then Lemma 3 implies that μ_ω must be a version of $P(g_t^{-1}|\mathcal{C})$, for all $t \in T^*$. It is also clear that $\lambda(T^*) = 1$.

Next, take any $t' \in T$, $B' \in \mathcal{B}$ and $C \in \mathcal{C}$. Because equation (10) holds when $A = C \cap g_{t'}^{-1}(B')$, it follows that

$$P(C \cap g_{t'}^{-1}(B') \cap g_t^{-1}(B)) = \int_C 1_{g_{t'}^{-1}(B')} \mu_\omega(B) dP$$

for λ-a.e. $t \in T$. By Lemma 1, the mapping $\omega \mapsto \mu_\omega(B)$ is $\mathcal{C}$-measurable. Also, $\mu_\omega(B') = E(1_{g_{t'}^{-1}(B')}|\mathcal{C})$ for λ-a.e. $t' \in T$. So $\int_C 1_{g_{t'}^{-1}(B')} \mu_\omega(B) dP = \int_C \mu_\omega(B') \mu_\omega(B) dP$ for λ-a.e. $t' \in T$. Thus, given any $C \in \mathcal{C}$ and any $B, B' \in \mathcal{B}$, for λ-a.e. $t \in T$, one has

$$P(C \cap (g_t, g_{t'})^{-1}(B \times B')) = \int_C (\mu_\omega \times \mu_\omega)(B \times B')\, dP \tag{11}$$

for λ-a.e. $t' \in T$. By the hypothesis that the mapping $(t, t') \mapsto P(A \cap g_t^{-1}(B) \cap g_{t'}^{-1}(B'))$ is $\mathcal{T} \otimes \mathcal{T}$-measurable for each $A \in \mathcal{A}$ and $B, B' \in \mathcal{B}$, it follows that equation (11) must hold on a $\mathcal{T} \otimes \mathcal{T}$-measurable set. But then the Fubini Theorem implies that for each $C \in \mathcal{C}$ and $B, B' \in \mathcal{B}$, equation (11) must hold $\lambda \times \lambda$-a.e. in $T \times T$.

In particular, for each triple (m, m', n) of positive integers, there exists a set $K_{mm'n} \in \mathcal{T} \otimes \mathcal{T}$ with $(\lambda \times \lambda)(K_{mm'n}) = 1$ such that for all $(t, t') \in K_{mm'n}$, Equation (11) holds with $C = C_n$, $B = B_m$ and $B' = B_{m'}$. But then

Equation (11) holds for all pairs (t, t') in the intersection $K^* := \cap_{m=1}^\infty \cap_{m'=1}^\infty \cap_{n=1}^\infty K_{mm'n}$, which is a set whose measure w.r.t. $(\lambda \times \lambda)$ is 1.

Hence, for each $(t, t') \in K^*$, Equation (11) holds for all $C \in \mathcal{C}^\pi$ and $B, B' \in \mathcal{B}^\pi$. Since $\{B \times B' : B, B' \in \mathcal{B}^\pi\}$ is a π-system that generates $\mathcal{B} \otimes \mathcal{B}$, Lemma 3 implies that $\mu_\omega \times \mu_\omega$ must be a version of $P((g_t, g_{t'})^{-1} | \mathcal{C})$. Putting $B' = X$ in (11) reduces it to $P(C \cap g_t^{-1}(B)) = \int_C \mu_\omega(B)\, dP$, so Lemma 3 implies similarly that μ_ω must be a version of $P(g_t^{-1} | \mathcal{C})$. The rest follows from Lemma 6. □

6.2 Proof that (2) $\Longrightarrow$ (4)

Let $f : T \times X \to \mathbb{R}$ be a $\mathcal{T} \otimes \mathcal{B}$-measurable function with the property that the mapping $t \mapsto E(f_t^2(g_t))$ defines an integrable function on T — i.e., $\int_T \int_\Omega f_t^2(g_t(\omega))\, dP\, d\lambda$ exists. The following Lemma 10 proves a strengthened version of Condition 3, with the indicator function 1_J of any set $J \in \mathcal{T} \otimes \mathcal{B}$ replaced by f. It is somewhat similar to the classical law of large numbers for a sequence of i.i.d. random variables taking values in a Banach space — see [32] and the detailed references in [20]. Lemma 10, however, involves a randomly selected sequence of real-valued square-integrable random variables that converges almost surely on Ω, whereas the conclusion of the corresponding law of large numbers in a Banach space states only convergence in the Banach space norm (the L_2-norm in this case).

The proof of Lemma 10 relies on the following elementary technical result.

Lemma 9 *Suppose that $\{a_i\}_{i=1}^\infty$ is a sequence of non-negative real numbers for which $\frac{1}{n} \sum_{i=1}^n a_i$ converges to the finite limit a as $n \to \infty$. Then the series $\sum_{n=1}^\infty a_n \frac{\log^2 n}{n^2}$ is convergent.*

Proof We group terms and write $\sum_{n=1}^\infty a_n \frac{\log^2 n}{n^2}$ as $\sum_{m=1}^\infty b_m$, where each $b_m := \sum_{n=2^{m-1}}^{2^m - 1} a_n \frac{\log^2 n}{n^2}$. Clearly, it is enough to show that $\sum_{m=1}^\infty b_m$ converges. But

$$b_m \le \sum_{n=2^{m-1}}^{2^m - 1} a_n \frac{\log^2 2^m}{(2^{m-1})^2} \le 4^{1-m} (m \log 2)^2 \sum_{n=1}^{2^m} a_n = c_m\, m^2\, 2^{-m}$$

where

$$c_m := 4 \log^2 2 \left(2^{-m} \sum_{n=1}^{2^m} a_n \right) \to 4a \log^2 2$$

as $m \to \infty$. This is enough to guarantee that $b_m \le c\, 2^{-m/2}$ for a suitable value of the constant c, so the series $\sum_{m=1}^\infty b_m$ does converge. □

Lemma 10 *Suppose that the process g is essentially i.i.d. conditioned on $\mathcal{C}$, with $P(g_t^{-1}|\mathcal{C}) = \mu_\omega$ for λ-a.e. $t \in T$. Let $f : T \times X \to \mathbb{R}$ be any $\mathcal{T} \otimes \mathcal{B}$-measurable function with $\int_T [\int_\Omega f_t^2(g_t(\omega))\, dP]\, d\lambda < \infty$. Then, for λ^∞-a.e. sequence $t^\infty \in T^\infty$, one has*

$$\frac{1}{n}\sum_{i=1}^{n} f(t_i, g(t_i,\omega)) \underset{P-a.s.}{\longrightarrow} \int_{T\times X} f(t,x)\, d(\lambda \times \mu_\omega). \tag{12}$$

Proof Given the specified function f, for each $t \in T$ and $\omega \in \Omega$, define

$$\psi_t(\omega) := f_t(g_t(\omega));\ \varphi(t,\omega) := \int_X f_t(x) d\mu_\omega(x);\ h_t(\omega) := \psi_t(\omega) - \varphi_t(\omega).$$

By hypothesis, ψ_t is P-square-integrable (and so P-integrable) for λ-a.e. $t \in T$. Since $P(g_t^{-1}|\mathcal{C}) = \mu$, Lemma 2 implies that for λ-a.e. $t \in T$, $E(\psi_t|\mathcal{C})(\omega) = \int_X f_t(x) d\mu_\omega(x) = \varphi(t,\omega)$ for P-a.e. $\omega \in \Omega$.

Now we apply the conditional version of Jensen's inequality for convex functions — see, for example, Theorem 10.2.7 of [12], or p. 225 of [13]. This inequality implies that $E(\varphi_t^2) = E[E(\psi_t|\mathcal{C})]^2 \le E(\psi_t^2)$ for λ-a.e. $t \in T$. Because ψ_t is square-integrable on $(\Omega, \mathcal{A}, P)$ for λ-a.e. $t \in T$, and $E(\psi_t^2)$ is integrable on $(T, \mathcal{T}, \lambda)$, it is easy to see that φ_t and h_t have the same two properties. It follows from the joint measurability of f and the measurability of $\omega \mapsto \mu_\omega \in \mathcal{M}(X, \mathcal{B})$ that φ is $\mathcal{T} \otimes \mathcal{A}$-measurable. Then, because ψ_t is square-integrable on $(\Omega, \mathcal{A}, P)$ for λ-a.e. $t \in T$, and because $E(\psi_t^2)$ is integrable on $(T, \mathcal{T}, \lambda)$, the Fubini Theorem implies that φ is square integrable w.r.t $\lambda \times P$.

Since g is essentially i.i.d. conditioned on $\mathcal{C}$, we have $P((g_t, g_{t'})^{-1}|\mathcal{C})(\omega) = \mu_\omega \times \mu_\omega$ for $\lambda \times \lambda$-a.e. $(t,t') \in T \times T$. Using Lemma 2 again,

$$E(\psi_t \psi_{t'}|\mathcal{C})(\omega) = \int_{X\times X} f_t(x) f_{t'}(y)\, d(\mu_\omega(x) \times \mu_\omega(y)) = \int_X f_t\, d\mu_\omega \int_X f_{t'}\, d\mu_\omega.$$

From the definition of h_t, it is easy to see that

$$E(h_t h_{t'}|\mathcal{C}) = E(\psi_t \psi_{t'}|\mathcal{C}) - E(\psi_t|\mathcal{C}) E(\psi_{t'}|\mathcal{C}) = 0$$

holds for $\lambda \times \lambda$-a.e. $(t,t') \in T \times T$. Hence there exists a $\mathcal{T} \otimes \mathcal{T}$-measurable set D such that $(\lambda \times \lambda)(D) = 1$ and $E(h_t h_{t'}) = 0$ for all $(t,t') \in D$. Now define D^* as the set of all sequences $t^\infty = (t_i)_{i=1}^\infty \in T^\infty$ such that $(t_i, t_j) \in D$ for all $i, j \in \mathbb{N}$. An elementary argument shows that $\lambda^\infty(D^*) = 1$. Hence, for all $t^\infty \in D^*$, the random variables $(h_{t_i})_{i=1}^\infty$ are mutually orthogonal.

Now, since $\int_T E(h_t^2) d\lambda < \infty$, the usual strong law of large numbers implies that for λ^∞-a.e. $t^\infty \in T^\infty$, $\frac{1}{n}\sum_{i=1}^n E(h_{t_i}^2)$ converges to $\int_T E(h_t^2) d\lambda$ as $n \to \infty$. Because $\frac{1}{n}\sum_{i=1}^n E(h_{t_i}^2)$ converges, Lemma 9 implies that the moment condition $\sum_{n=1}^\infty \frac{1}{n^2} \log^2 n\, E(h_{t_n}^2) < \infty$ of the strong law of large numbers in [11], Theorem 5.2, p. 158 is satisfied for λ^∞-a.e. $t^\infty \in T^\infty$. That result therefore applies to the random variables $(h_{t_i})_{i=1}^\infty$, because they are mutually orthogonal for λ^∞-a.e. $t^\infty \in T^\infty$. It implies that for λ^∞-a.e. $t^\infty \in T^\infty$, one has

$$\frac{1}{n}\sum_{i=1}^{n} h_{t_i}(\omega) \xrightarrow[P-\text{a.s.}]{} 0. \tag{13}$$

Because φ is square-integrable (and so integrable) on the product space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$, the Fubini Theorem implies that φ_ω is λ-integrable on T, for P-a.e. $\omega \in \Omega$; the usual strong law of large numbers then implies that for λ^∞-a.e. $t^\infty \in T^\infty$, $\frac{1}{n}\sum_{i=1}^{n} \varphi_\omega(t_i)$ converges to $\int_T \varphi_\omega(t) d\lambda(t)$. Using the Fubini theorem yet again, the relevant null sets can be interchanged, and so for λ^∞-a.e. $t^\infty \in T^\infty$,

$$\frac{1}{n}\sum_{i=1}^{n} \varphi_{t_i}(\omega) \xrightarrow[P-\text{a.s.}]{} \int_T \varphi_\omega(t) d\lambda(t). \tag{14}$$

By Equations (13) and (14), we obtain that for λ^∞-a.e. $t^\infty \in T^\infty$,

$$\frac{1}{n}\sum_{i=1}^{n} \psi_{t_i}(\omega) = \frac{1}{n}\sum_{i=1}^{n} h_{t_i}(\omega) + \frac{1}{n}\sum_{i=1}^{n} \varphi_{t_i}(\omega) \xrightarrow[P-\text{a.s.}]{} \int_T \varphi_\omega(t) d\lambda(t). \tag{15}$$

But $\int_T \varphi_\omega(t) d\lambda(t) = \int_T [\int_X f_t(x) d\mu_\omega(x)] d\lambda = \int_{T \times X} f(t,x)\, d(\lambda \times \mu_\omega)$, so the result follows from (15). □

For each $x \in X$, let δ_x denote the degenerate probability measure attaching probability 1 to x. Then, for each single random draw $t^\infty \in T^\infty$ and each $\omega \in \Omega$, each measure $\mu^n_{t^\infty,\omega} := \frac{1}{n}\sum_{i=1}^{n} \delta_{g(t_i,\omega)}$ $(n = 1, 2, \ldots)$ is the empirical distribution of x given the n observations $g(t_i, \omega)$ $(i = 1, 2, \ldots, n)$. The following corollary says that μ_ω is identified as the (almost sure) weak limit of $\mu^n_{t^\infty,\omega}$.

Corollary 2 *For λ^∞-a.e. sequence $t^\infty \in T^\infty$, the empirical distribution $\mu^n_{t^\infty,\omega}$ converges weakly to μ_ω, for P-almost all $\omega \in \Omega$.*

Proof We apply Theorem 6.6 on p. 47 of [26]. Because X is a Polish space, so in particular a separable metric space, this theorem implies that there exist an equivalent metric on X and a sequence of bounded and uniformly continuous functions $\varphi_m : X \to \mathbb{R}$ $(m = 1, 2, \ldots)$ with the property that, for each $t^\infty \in T^\infty$ and each $\omega \in \Omega$, the distribution $\mu^n_{t^\infty,\omega}$ converges weakly to μ_ω if and only if for each $m = 1, 2, \ldots$, the mean $\int_X \varphi_m(x)\, d\mu^n_{t^\infty,\omega}$ converges to $\int_X \varphi_m(x)\, d\mu_\omega$ as $n \to \infty$.

For each fixed $m = 1, 2, \ldots$, because φ_m is measurable and bounded, the definition of $\mu^n_{t^\infty,\omega}$ and Lemma 10 together imply that for λ^∞-a.e. sequence $t^\infty \in T^\infty$,

$$\int_X \varphi_m(x)\, d\mu^n_{t^\infty,\omega} = \frac{1}{n}\sum_{i=1}^{n} \varphi_m(g(t_i,\omega)) \xrightarrow[P-\text{a.s.}]{} \int_X \varphi_m(x)\, d\mu_\omega. \tag{16}$$

Because one can group together countably many λ^∞-null sets, there exists a subset T_1^∞ of T^∞ with $\lambda^\infty(T_1^\infty) = 1$ such that for each sequence $t^\infty \in T_1^\infty$, Equation (16) holds for all m simultaneously. Consider any sequence $t^\infty \in T_1^\infty$. Again, because one can group together countably many P-null sets, for P-almost all $\omega \in \Omega$ one has $\int_X \varphi_m(x)\, d\mu^n_{t^\infty,\omega} \to \int_X \varphi_m(x)\, d\mu_\omega$ for all m simultaneously. This implies that for each sequence $t^\infty \in T_1^\infty$, the sufficient condition for $\mu^n_{t^\infty,\omega}$ to converge weakly to μ_ω is satisfied for P-almost all $\omega \in \Omega$. $\square$

6.3 Proof that (4) $\Longrightarrow$ (1)

Of course, (4) $\Longrightarrow$ (3) is obvious. To complete the proof of Theorem 1, therefore, we only need to prove the following lemma.

Lemma 11 *Suppose that for each $S \in \mathcal{T}$, $B \in \mathcal{B}$, and for λ^∞-a.e. sequence $t^\infty \in T^\infty$, one has*

$$\frac{1}{n}\sum_{i=1}^{n} 1_S(t_i)\, 1_B(g(t_i,\omega)) \underset{P-a.s.}{\longrightarrow} \lambda(S)\, \mu_\omega(B) \tag{17}$$

Then for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, one has $P(A \cap g_t^{-1}(B)) = \int_A \mu_\omega(B)\, dP$ for λ-a.e. $t \in T$.

Proof Integrating (17) w.r.t. ω over any measurable set $A \in \mathcal{A}$ yields the result that, for λ^∞-a.e. $t^\infty \in T^\infty$ one has

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n} 1_S(t_i) \int_A 1_B(g(t_i,\omega))\, dP \\ &= \frac{1}{n}\sum_{i=1}^{n} 1_S(t_i)\, P(A \cap g_{t_i}^{-1}(B)) \to \lambda(S) \int_A \mu_\omega(B)\, dP \end{aligned} \tag{18}$$

Now, the hypothesis that probabilities are pairwise measurable clearly implies that $t \mapsto P(A \cap g_t^{-1}(B))$ is $\mathcal{T}$-measurable. It follows that for any $S \in \mathcal{T}$, the mapping $t \mapsto 1_S(t)\, P(A \cap g_t^{-1}(B))$ is also $\mathcal{T}$-measurable. By the usual strong law of large numbers, therefore,

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} 1_S(t_i) P(A \cap g_{t_i}^{-1}(B)) &\to \int_T 1_S(t)\, P(A \cap g_t^{-1}(B)) d\lambda \\ &= \int_S P(A \cap g_t^{-1}(B)) d\lambda \end{aligned} \tag{19}$$

for λ^∞-a.e. $t^\infty \in T^\infty$. Because the two limits in (18) and (19) must be equal,

$$\int_S P(A \cap g_t^{-1}(B))\, d\lambda = \lambda(S) \int_A \mu_\omega(B)\, dP = \int_S \left[\int_A \mu_\omega(B)\, dP \right] d\lambda$$

for all $S \in \mathcal{T}$. By the essential uniqueness of the Radon–Nikodym derivative, it follows that

$$P(A \cap g_t^{-1}(B)) = \int_A \mu_\omega(B)\, dP$$

for λ-a.e. $t \in T$. □

7 Concluding assessment

In mathematical economics, following the pioneering contributions of Vickrey [33] and Aumann [2], [3] respectively, it has become common to consider continuous density functions of relevant consumer characteristics, or more general economic models with a continuum of agents. These, of course, are mathematical abstractions which cannot hold exactly in any actual economy, with a finite set of agents. Nevertheless, they provide convenient approximations when used with appropriate care.

In models where agents face individual risk, the joint and sample measurability problems described in the introduction have made it difficult to provide rigorous foundations for the intuitively appealing idea that, with a continuum of agents, some version of the law of large numbers should hold exactly rather than approximately. The earlier work in [27], [28] and [30] shows that this obstacle can be overcome in an extended product measure-theoretic framework with the usual Fubini property. The only known examples of such a framework involve Loeb product spaces.

As mentioned in the introduction, if one adopts the asymptotic point of view by simply taking a randomly drawn sequence from a continuum of i.i.d. random variables, then the classical law of large numbers trivially applies to this sequence. This paper moves beyond this law to consider a large economy modeled by a continuum of essentially pairwise exchangeable random variables.[13] Our parameter space $(T, \mathcal{T}, \lambda)$ can be any atomless probability space, including the Lebesgue unit interval and hyperfinite Loeb spaces. Proposition 2 shows that the joint measurability problem cannot be avoided unless almost all agents' shocks are identical and thus perfectly correlated. Nevertheless, even without joint or sample measurability, it is shown that the "almost everywhere" or "essential" versions of the symmetry, pairwise exchangeability and conditional i.i.d. properties are all equivalent.

An important issue in macroeconomics is to devise a general mathematical framework allowing individual agents to face random idiosyncratic shocks which are independent when conditioned on suitably constructed random macroeconomic states. Ideally, the macroeconomic states should have a simple interpretation, and even be identifiable empirically. This paper shows how Monte Carlo simulation can achieve that purpose, with the macroeconomic states in the symmetric case being just the weak limit of

[13] We believe that even this type of symmetry condition can be greatly relaxed, as we plan to discuss in later work.

the empirical distributions obtained from a single random draw $t^\infty \in T^\infty$, as shown in Corollary 2.

This accords with many existing macroeconomic models, including natural extensions to a continuum of agents of the models involving simple independently distributed stationary (SIDS) processes devised by Nielsen [25], which appear in a particularly simple and appealing class of rational belief equilibria of the kind considered by Mordecai Kurz and various collaborators — see especially [22] and [23]. Indeed, in such a continuum extension, a key part of the macroeconomic state would be the history of what proportions of agents have optimistic or pessimistic beliefs of various degrees at different times.

Finally, as a by-product of our work, we have shown that the fundamental probabilistic concept of (essential) independence constitutes a necessary condition for the classical sequential law of large numbers to hold. This converse result is entirely new in the extensive mathematical literature on the subject.

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