Efficiency with Non-Convexities: Extending the "Scandinavian Consensus" Approaches

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Abstract

There are two distinct "Scandinavian consensus" approaches to public good supply, both based on agents' willingness to pay. A Wicksell–Foley public competitive equilibrium arises from a negative consensus in which no change of public environment, together with associated taxes and subsidies which finance it, will be unanimously approved. Alternatively, in a Lindahl or valuation equilibrium, charges for the public environment induce a positive consensus. To allow general non-convexities to be regarded as aspects of the public environment, we extend recent generalizations of these equilibrium notions and prove counterparts to both the usual fundamental efficiency theorems of welfare economics.

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Efficiency with Non-Convexities

1. Introduction and Outline

1.1. Non-Convexities and the Public Environment

A modern mixed economy can be regarded as combining a private market sector with a public non-market sector. In the private market sector, individual economic agents make decisions within their private feasible sets. These private agents take as given certain important variables which are determined outside the market mechanism, often as a result of deliberate public policy decisions. Examples of such non-market variables include prevailing social rules like the legal system and especially the assignment of property rights. They also include private goods provided by the public sector, such as many transport, health and education services. Often, what matters here is the quality rather than the quantity of these services. Other non-market variables can be used to describe the regulation of private economic activity through quotas, quality standards, legislation affecting health and safety at work, etc. The working of the tax and benefit system in the economy is yet another kind of non-market variable. Finally, many externalities and environmental concerns involve non-market variables, even if the rights or duties to create externalities are allocated through the price mechanism — e.g., through a market for pollution licences, or allocating the contract for providing an unprofitable but socially desirable bus or rail service to whichever private firm demands the lowest subsidy.

All such non-market variables constitute what we choose to call the *public environment*, or simply the *environment*. This is a very broad concept allowing many different economic problems to be treated within one unified framework. The environment in this sense will be treated as a public good, that is as a collection of variables which are common to all agents. Needless to say, the environment may in turn affect agents' feasible sets and objectives. Devising a good procedure which determines each aspect of the public environment, together with the taxes needed to finance that environment, is obviously one of the key tasks of economic policy makers.

In order to ecompass many different situations, our mathematical framework follows Mas-Colell (1980) in allowing the vector z of variables describing the environment to range over an abstract set Z with no special structure. In particular, Z need not be convex.

This framework allows us to discuss, in principle, not only non-convexities in public sector decisions, but also issues such as whether to allow production by private firms with fixed set-up costs. The idea of treating what may be private non-convexities in this way appears to be due to Malinvaud (1969, 1972) and to an unpublished Ph. D. thesis due to Beato. See also Dierker (1986) and Laffont (1988).

Readers may recall that classical writers like Say (1826), Dupuit (1844), or Hotelling (1938) discussed large projects such as those involving roads, bridges, canals, or railways. Whether or not the constructor and/or the operator is privately owned, such projects inevitably have many of the features of a public good. In part, this is because they create pecuniary externalities in the form of modifications to the price system as a whole, especially in the geographical vicinity of any such project. For this reason, a major issue of public policy is to create guidelines for determining the criteria under which each such potential project is to be accepted or rejected, and how the project is to be financed if it is accepted. In fact, whether a private firm should incur significant set-up costs is virtually always a public policy issue, even if it is usually not recognized as such. This is because it shares the key features of a decision affecting public goods or the public environment in some more conventional sense.

With this background in mind, the main concern of the paper will be to characterize Pareto efficient allocations in such a mixed economy, when for each given public environment, including variables describing non-convexities, agents trade private goods competitively. In order to do so we shall look for equilibrium concepts for which both the efficiency theorems of welfare economics are true — that is, any equilibrium allocation should be Pareto efficient, and conversely (under suitable assumptions). In this respect, the much discussed marginal cost pricing rules fail because, even with suitable lump-sum redistribution of initial wealth, not all marginal cost pricing equilibria need be Pareto efficient — see, for example, the recent discussions by Quinzii (1992) and Villar (1994, 1996). Our characterization will involve suitably revised versions of both the Wicksell (1896) and the Lindahl (1919) approaches to the efficient provision of public goods. More specifically, we shall consider appropriate revisions of the equilibrium notions which more modern economic theorists have created in order to capture their ideas. These notions may be regarded as further extensions of the “Scandinavian consensus” — to use the evocative term due to Bergstrom (1970), who analysed consumption externalities using a development of Lindahl’s approach.

1.2. Two Kinds of Consensus

In fact, the literature on public goods presents us with two main approaches to the problem of achieving a Pareto efficient allocation. The first harks back to Wicksell (1896), but was originally formalized in modern mathematical terms by Foley (1967). This approach tends to regard the choice of public goods z as an essentially political matter, about which the economist has little to say. This is the kind of allocation which Foley called “publicly competitive”, and which Malinvaud (1969, 1972) called, perhaps more appropriately, a “politico-economic” equilibrium. The Wicksell–Foley idea is to have either the community of agents, or its representatives in government, draw up proposals for both public good production and taxes to finance this production. These are “public sector proposals.” Then any public sector proposal can be amended if and only if each consumer in the economy, and each producer, favours the amendment. A consumer will favour the amendment if, taking present prices for private goods as given, the change in the public goods which are provided gives a net benefit exceeding the net cost of any extra taxes that have to be paid. A firm will favour the amendment if the change in the public goods which are provided, including those which it might be called on to produce, allows it to cover the (net) cost of any extra taxes it has to pay from the extra net profit it makes at fixed prices. An equilibrium results from a negative consensus, in which agents are unable to agree unanimously on how to change the public environment.

The second main approach to the problem of achieving Pareto efficiency with public goods is named after Lindahl (1919). It requires all consumers and private producers to pay a “Lindahl price” for each public good according to the marginal benefits which they receive from it. Each public good is produced so that the total marginal benefit to all private consumers and producers is equal to the marginal cost of providing it. In equilibrium the Lindahl prices must be chosen so as to reach a positive consensus, in which all agents agree that the same public environment is optimal, given their budget constraints or profit functions.

When non-convexities are involved, however, setting prices equal to marginal benefits is clearly going to be insufficient, in general. So we adapt an idea due to Mas-Colell (1980) and allow a valuation scheme with non-linear Lindahl pricing — see also Vega-Redondo (1987), as well as Diamantaras and Gilles (1996), Diamantaras, Gilles and Scotchmer (1996). This leads us to consider what Mas-Colell calls a *valuation equilibrium*, which is defined as a price vector, a tax system, a feasible allocation, and a public environment such that: (a) each consumer's equilibrium combination of a private net trade vector together with the public environment is weakly preferred to any other such combination which is affordable, given the equilibrium prices and the non-linear tax system; (b) each firm's combination of a net output vector for private goods together with the public environment is chosen to maximize profit over its production set, given the equilibrium prices and the non-linear tax system; and (c) aggregate net tax payments (i.e., taxes less subsidies) are zero.[1]

1.3. Outline of Paper

In the rest of the paper, Section 2 describes our model. It is a "conditionally convex" economy in the sense that, for each fixed public environment $z \in Z$, there is a standard Arrow–Debreu private good economy satisfying the usual convexity and continuity conditions.

Thereafter, Section 3 considers publicly competitive equilibria, using an extension of Foley's definition. In Foley's formulation, private agents are implicitly assumed to be "myopic" in the sense that they ignore how the equilibrium prices of private goods depend on the choice of public environment. This is an instance of price-taking behaviour in which agents neglect the influence of even collective decisions upon private good prices. Of course, when there is only one private good, as in Mas-Colell (1980) and many other papers, this is not an issue. Also, in the absence of non-convexities, one can usually disregard the effect of any marginal change in the public environment on the equilibrium prices of private goods.

By contrast, we assume that all agents see how the change in public environment passes an appropriate cost–benefit test when one considers a suitable new conditional equilibrium price vector. In this sense, agents are assumed to be "far-sighted." This plays an important

[1] In a private economy where some production sets may be non-convex, Brown and Heal (1980) consider interesting decentralizations of Pareto efficient allocations by means of non-linear "value functions". These functions, however, are restricted to be homogeneous of degree one, which only makes sense when the domain is a linear space. In addition, their equilibria need not be Pareto efficient, so their equilibrium concept does not characterize efficient allocations.

role in finding counterparts to the usual efficiency theorems of welfare economics when there are many private goods and also non-convexities associated with the public environment.

Section 4 turns towards the Lindahl approach, and — like Diamantaras and Gilles (1996), Diamantaras, Gilles and Scotchmer (1996) — looks for a simple generalization to economies with many private goods of those results due to Mas-Colell (1980) which characterize Pareto efficient allocations as valuation equilibria.[2] Unlike previous writers, however, we allow the non-linear valuation of the public environment to depend also on prices for private commodities. This allows Mas-Colell's results to be extended to economies with many private commodities under somewhat less restrictive conditions than those imposed by Diamantaras *et al.*

Finally, Section 5 summarizes the main results. It also contains a brief concluding discussion of iterative adjustment procedures and of incentive constraints.

2. A Conditionally Convex Economy

2.1. Assumptions

Consider an economy with a finite set G of private commodities, a finite set I of individual consumers, and a finite set J of producers. In addition, suppose there is an abstract set Z whose members are vectors of those variables which define the *environment.* Each agent's feasible set and objectives may be affected by the values taken by the vector $z \in Z$. No particular structure will be postulated on the set Z, though it may be thought of as a subset of $\Re^k$, for some k. We assume that some kind of public agency or *public sector* determines these variables. We also assume that the public sector may affect consumers' budget sets and firms' profit functions via taxes and subsidies. Even though the tax system itself can be thought of as part of the environment, we find it more convenient to treat these variables separately.

We assume that the economy is *conditionally convex*, meaning that the public environment $z \in Z$ is able to capture all the relevant non-convexities in the economy, in the following sense. Each individual $i \in I$ is assumed to have:

[2] Attempts to generalize results on the core and on adjustment procedures are left for later work. See also Diamantaras and Gilles (1996).

(1) a feasible set $\tilde{X}^i \subset \Re^G \times Z$ such that, for each $z \in Z$, the *conditionally feasible set*

$$X^i(z) := \{ x^i \in \Re^G \mid (x^i, z) \in \tilde{X}^i \}$$

of private good net trade vectors is convex and closed; it will be assumed in addition that $X^i(z)$ is bounded below by a vector $\underline{x}^i(z)$ with the property that $x \in X^i(z)$ implies $x \geqq \underline{x}^i(z)$;

(2) a (complete and transitive) preference ordering R^i on $\tilde{X}^i$ such that, for each $z \in Z$, the *conditional preference ordering* $R^i(z)$ defined by

$$\forall x^i, \bar{x}^i \in X^i(z) : x^i \; R^i(z) \; \bar{x}^i \iff (x^i, z) \; R^i \; (\bar{x}^i, z)$$

is convex, continuous, and locally non-satiated.

In addition, each producer $j \in J$ is assumed to have:

(3) a production set $\tilde{Y}^j \subset \Re^G \times Z$ such that, for each $z \in Z$, the *conditional production set*

$$Y^j(z) := \{ y^j \in \Re^G \mid (y^j, z) \in \tilde{Y}^j \}$$

is convex and closed.

Essentially the above axioms say that, for any given value of the environmental variables, the resulting conditional economy is standard. Observe that these axioms involve no restriction on the set Z, which may therefore contain all kinds of variables. So our formulation allows many different kinds of non-convexity.

Next, for each individual $i \in I$, for each pair $(\hat{x}^i, \hat{z}) \in \tilde{X}^i$, and any alternative public environment $z \in Z$, define the two sets

$$P^i(\hat{x}^i, \hat{z}; z) := \{ x^i \in X^i(z) \mid (x^i, z) \; P^i \; (\hat{x}^i, \hat{z}) \}$$
$$R^i(\hat{x}^i, \hat{z}; z) := \{ x^i \in X^i(z) \mid (x^i, z) \; R^i \; (\hat{x}^i, \hat{z}) \}$$

Both are upper preference sets in $\Re^G$. Because preferences are locally non-satiated in private goods, note that $P^i(\hat{x}^i, \hat{z}; z)$ is non-empty whenever $R^i(\hat{x}^i, \hat{z}; z)$ is.

In the following, let $\mathbf{X}^I(z)$ and $\mathbf{Y}^J(z)$ denote the Cartesian products $\prod_{i \in I} X^i(z)$ and $\prod_{j \in J} Y^j(z)$ respectively.

This paper is concerned with conditions under which a particular feasible allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ is an equilibrium. When considering alternative public environments, it will lose no generality to restrict attention to the subset $\hat{Z}$ of Z whose members satisfy the requirement that, for each $i \in I$, the conditional weak preference set $R^i(\hat{x}^i, \hat{z}; z)$ is non-empty. This excludes those public environments z which are so bad for some individual $i \in I$ that no choice of net trade vector could possibly compensate i for upsetting the *status quo* $(\hat{x}^i, \hat{z}) \in \tilde{X}^i$. Because of local non-satiation, the corresponding conditional strict preference set $P^i(\hat{x}^i, \hat{z}; z)$ must also be non-empty for each $i \in I$. Note that $\hat{Z}$ consists precisely of those $z \in Z$ which would allow consumers to reach a weakly Pareto superior allocation provided that the economy were sufficiently productive.

2.2. Restricted Profit and Compensation Functions

For every price vector $p \neq 0$ and every public environment z, define the (restricted) profit function of each private producer $j \in J$ in the obvious way by

$$\pi^j(p, z) := \sup_y \{ p\, y \mid y \in Y^j(z) \}$$

Note that $+\infty$ is admitted as a possible value of a profit function, but this causes no difficulty.

This paper will consider conditions for particular allocations $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ to be decentralizable equilibria of various kinds. These conditions will involve measures of anticipated consumer and producer benefit associated with deviations from the equilibrium allocation. To this end, given the particular allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ and any pair (p, z) with $p \neq 0$ and $z \in Z$, it is useful to introduce the notation

$$\hat{e}^i(p, z) := \min_{x^i} \{ p\, x^i \mid x^i \in R^i(\hat{x}^i, \hat{z}; z) \}$$

for each consumer i's *compensation function.* This is the minimum expenditure on private goods needed to ensure that individual i is no worse off than at $(\hat{x}^i, \hat{z})$, given the price vector p and the alternative public environment z. The assumptions of Section 2.1 imply that $\hat{e}^i(p, z)$ is well-defined and finite whenever $p > 0$ and $R^i(\hat{x}^i, \hat{z}; z)$ is non-empty, as it must be whenever $z \in \hat{Z}$. Provided that $R^i(\hat{x}^i, \hat{z}; z)$ is non-empty, we assume that $\hat{e}^i(p, z)$ is well defined, even when $p \not> 0$. Finally, define

$$\hat{S}(p, z) := \sum_{j \in J} \pi^j(p, z) - \sum_{i \in I} \hat{e}^i(p, z)$$

This *aggregate net benefit function* measures the surplus by which aggregate profit exceeds the minimum aggregate wealth needed to ensure that no consumer i is worse off than with $(\hat{x}^i, \hat{z})$. When $\hat{S}(p, z) > 0$, then moving to the new public environment z effectively passes a cost–benefit test indicating a potential Pareto improvement — provided that, after this move combined with suitable lump-sum wealth redistribution, private good markets will clear at the price vector p.

2.3. Private Good Competitive Allocations

The allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ is said to be *private good competitive* at the non-zero price vector $p \in \Re^G \setminus \{0\}$ provided that:

(1) for all $i \in I$, the net trade vector $\hat{x}^i$ maximizes $R^i(\hat{z})$ subject to $x^i \in X^i(\hat{z})$ and $p\, x^i \leq p\, \hat{x}^i$;

(2) for all $j \in J$, the net output vector $\hat{y}^j$ maximizes $p\, y^j$ subject to $y^j \in Y^j(\hat{z})$;

(3) $\sum_{i \in I} \hat{x}^i = \sum_{j \in J} \hat{y}^j$.

Thus, all agents treat the public environment $\hat{z}$ as fixed. Also, each consumer $i \in I$ maximizes the conditional preference ordering $R^i(\hat{z})$ given the budget constraint $p\, x^i \leq w^i$, where $w^i := p\, \hat{x}^i$ is a level of wealth just large enough for i to afford $\hat{x}^i$. And each firm $j \in J$ maximizes profits $p\, y^j$ over its conditional production set $Y^j(\hat{z})$. Finally, (3) is the resource balance constraint, which obviously entails the budget balance constraint $\sum_{i \in I} p\, \hat{x}^i = \sum_{j \in J} p\, \hat{y}^j$. The right-hand side of this equation includes the aggregate net profits from the net output of private goods used in creating public goods. Such net profits are typically negative, of course. So (3) and the associated budget equation allow for the need to finance the inputs used in creating the public environment, as well as representing the distribution to individual consumers of the profits arising from producing private goods.

Say also that the allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ is *private good compensated competitive* at the price vector $p \neq 0$ provided that (1) above is replaced by:

(1′) for all $i \in I$, the net trade vector $\hat{x}^i$ minimizes $p\, x^i$ subject to $x^i \in X^i(\hat{z})$ and $x^i\; R^i(\hat{z})\; \hat{x}^i$;

whereas (2) and (3) are satisfied as before.

When preferences for private goods are locally non-satiated, such a private good competitive allocation will be *Pareto efficient given* $\hat{z}$ in the sense that there is no Pareto superior allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ with the same public environment. Conversely, given the standard assumptions set out above, any allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ which is Pareto efficient given $\hat{z}$ will also be private good compensated competitive at some price vector $p \neq 0$.

2.4. The Cheaper Point Lemma

As with the classical second efficiency theorem, it will only be shown here that a Pareto efficient allocation is some kind of *compensated* equilibrium. Additional assumptions such as those set out in Hammond (1993, 1998) are required to ensure that this compensated equilibrium is an uncompensated equilibrium. Indeed, the following result is a simple adaptation of one that is familiar in classical economic environments.

CHEAPER POINT LEMMA. *Suppose that the price vector $p \neq 0$ is such that, whenever $(x^i, z) \in \tilde{X}^i$ satisfies $(x^i, z)\ R^i\ (\hat{x}^i, \hat{z})$, then $p\, x^i \geq w^i$. Suppose too that there exists a "cheaper point" $\underline{x}^i \in X^i(z)$ with $p\, \underline{x}^i < w^i$. Then any $(x^i, z) \in \tilde{X}^i$ with $(x^i, z)\ P^i\ (\hat{x}^i, \hat{z})$ must satisfy $p\, x^i > w^i$.*

PROOF: Suppose that $(x^i, z) \in \tilde{X}^i$ with $(x^i, z)\ P^i\ (\hat{x}^i, \hat{z})$. The assumptions of Section 2.1 imply that $X^i(z)$ is convex and $R^i(z)$ is continuous. Accordingly, there must exist some small λ with $0 < \lambda < 1$ such that the point $x^i(\lambda) := x^i + \lambda\,(\underline{x}^i - x^i) \in X^i(z)$ and also $(x^i(\lambda), z)\ P^i\ (\hat{x}^i, \hat{z})$. Then $(x^i(\lambda), z) \in \tilde{X}^i$ and $(x^i(\lambda), z)\ R^i\ (\hat{x}^i, \hat{z})$, of course. So the hypothesis of the Lemma implies that

$$p\, x^i(\lambda) = p\,[x^i + \lambda\,(\underline{x}^i - x^i)] \geq w^i$$

This is equivalent to

$$(1 - \lambda)\, p\, x^i \geq w^i - \lambda\, p\, \underline{x}^i > (1 - \lambda)\, w^i$$

where the last strict inequality follows because $\lambda > 0$ and $p\, \underline{x}^i < w^i$. But then, dividing by $1 - \lambda$ which is also positive, we obtain $p\, x^i > w^i$. ∎

3. Far-Sighted Public Competitive Equilibrium

3.1. Generalized Public Sector Proposals

Following Wicksell's (1896) original insight, Foley (1967) considered public sector proposals in the form of a revised vector of public goods, together with taxes on consumers in order to finance the inputs needed to produce those public goods. Implicitly, however, his definition has private agents who are "myopic" in the sense that they treat the price vector p for private goods as independent of the public environment $z \in Z$. Also, he does not allow private producers. Greenberg (1975) has one aggregate private producer and allows more complex tax systems, but still has myopic agents.

As an obvious extension of the Wicksell–Foley approach, suppose that each consumer $i \in I$ faces a net tax $t^i(p)$ and each producer $j \in J$ faces a net subsidy $s^j(p)$, all of which depend on the price vector $p \neq 0$. Thus, there is a *tax/subsidy system* $(\mathbf{t}^I(p), \mathbf{s}^J(p)) := (\langle t^i(p)\rangle_{i\in I}, \langle s^j(p)\rangle_{j\in J})$. Moreover, suppose that each consumer i's net wealth is also a function $w^i(p)$ of p, and let $\mathbf{w}^I(p) := \langle w^i(p)\rangle_{i\in I}$ denote the economy's *wealth distribution rule.* Then a *generalized public sector proposal* is defined as a collection $(z, \mathbf{w}^I(p), \mathbf{t}^I(p), \mathbf{s}^J(p))$ satisfying the following conditions:

(a) all the functions $w^i(p)$, $t^i(p)$ and $s^j(p)$ are both continuous and homogeneous of degree one;

(b) $\sum_{i\in I} w^i(p) = \sum_{j\in J} [\pi^j(p, z) + s^j(p)]$;

(c) $\sum_{i\in I} t^i(p) = \sum_{j\in J} s^j(p)$;

(d) for all $i \in I$ and $p \neq 0$, one has $w^i(p) - t^i(p) > \hat{e}^i(p, z)$ whenever $\hat{S}(p, z) > 0$.

Of these conditions, (a) is intended to help ensure existence of competitive equilibrium in an obvious way. Evidently (b) and (c) require, respectively, the wealth distribution rule and the tax/subsidy system to be balanced.

Finally, whenever $\hat{S}(p, z) > 0$ and so the economy can afford to allow each consumer to spend more than $\hat{e}^i(p, z)$ on a private good net trade vector, condition (d) requires that they be allowed to do so. This will ensure that whenever p is a compensated equilibrium price vector in the conditional economy given z and this equilibrium p satisfies $\sum_{j\in J} \pi^j(p, z) > \sum_{i\in I} \hat{e}^i(p, z)$, then every consumer has a "cheaper point" in the conditionally feasible set $X^i(z)$ that lies below the budget hyperplane $p\, x^i + t^i(p) = w^i(p)$. By the Cheaper Point

Lemma of Section 2.4, it follows that the compensated equilibrium will be an ordinary or "uncompensated" equilibrium. Furthermore, because consumers maximize preferences in such an equilibrium and can find a $\tilde{x}^i$ with $(\tilde{x}^i, z)\ R^i\ (\hat{x}^i, \hat{z})$ such that $p\,\tilde{x}^i + t^i(p) < w^i(p)$, it must be true that each consumer's equilibrium net trade vector x^i satisfies $(x^i, z)\ P^i\ (\hat{x}^i, \hat{z})$. Essentially, condition (d) requires the wealth distribution rule, when combined with the tax rule, to convert potential Pareto improvements into actual ones.

3.2. Definitions

After these necessary preliminaries, consider the feasible allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ together with a price vector $\hat{p} \neq 0$. This combination is said to be a *far-sighted public competitive equilibrium* (or FSPCE) if:

(i) $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ is private good competitive at prices $\hat{p}$;

(ii) there is no generalized public sector proposal $(z, \mathbf{w}^I(p), \mathbf{t}^I(p), \mathbf{s}^J(p))$ permitting the existence of an associated feasible allocation $(\mathbf{x}^I, \mathbf{y}^J, z)$ which is private good competitive at a price vector $p^* \neq 0$ satisfying $p^*\,\hat{x}^i = w^i(p^*)$ for all $i \in I$, as well as passing the cost–benefit test $\hat{S}(p^*, z) > 0$.

According to (ii), therefore, there can be no alternative generalized public sector proposal, including the taxes needed to finance the altered public environment z, which allows the economy to reach an equilibrium at a price vector $p^* \neq 0$ for which the cost–benefit test $\hat{S}(p^*, z) > 0$ is passed.[3]

As Foley (1967) in particular admits, any such definition fails to specify what political process underlies the choice among the many public sector proposals which might satisfy condition (ii) when the economy is not at an FSPCE. The only assumption is that amendments will be made repeatedly until no further amendment which everybody favours can be found.

The combination $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z}, \hat{p})$ is a *compensated* FSPCE when condition (i) above is weakened to:

[3] An alternative equilibrium concept would modify (ii) to exclude directly any proposal allowing a private good competitive equilibrium in which all consumers attain preferred allocations within their respective budget sets, while each private firm makes no less profit after adjusting the net subsidy. That is, the public sector proposal cannot be amended in a way which is unanimously approved by all consumers and producers. Such an alternative concept may actually capture the idea of a negative consensus rather better. In any case, this modification would make the two efficiency results Theorems 1 and 2 below hold almost trivially.

(i′) $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ is private good compensated competitive at prices $\hat{p}$;

but (ii) remains the same as before.

Suppose that $(z, \mathbf{w}^I(p), \mathbf{t}^I(p), \mathbf{s}^J(p))$ is a generalized public sector proposal for which $(\mathbf{x}^I, \mathbf{y}^J, p^*)$ is a private good compensated competitive equilibrium satisfying $\hat{S}(p^*, z) > 0$. Then condition (d) of the previous definition in Section 3.1 implies that the cheaper point lemma of Section 2.4 is applicable. It follows that $(\mathbf{x}^I, \mathbf{y}^J, p^*)$ must be a private good (uncompensated) competitive equilibrium, and that $(x^i, z)\ P^i\ (\hat{x}^i, \hat{z})$ for all $i \in I$.

3.3. First Efficiency Theorem

First, add the plausible assumption that for each $z \in Z$, there is an upper bound $\bar{y}(z)$ for the set J of producers as a whole with the property that, whenever $\mathbf{y}^J \in \mathbf{Y}^J(z)$ satisfies $\sum_{j\in J} y^j \geqq \sum_{i\in I} \underline{x}^i(z)$, then $\sum_{j\in J} y^j \leqq \bar{y}(z)$. Essentially, this is the standard requirement that bounded inputs cannot generate unbounded outputs. In fact, given any public environment $z \in Z$, the combination of this extra assumption with those set out in Section 2.1 is sufficient to ensure compactness of the attainable set

$$A(z) := \{ (\mathbf{x}^I, y) \in \mathbf{X}^I(z) \times \sum_{j\in J} Y^j(z) \mid \sum_{i\in I} x^i = y \}$$

of all conditionally feasible combinations of consumer net trade vectors with an aggregate net output vector.

Consider any *status quo* feasible allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$. Recall that the definitions and assumptions of Section 2.1 already imply that, for all $i \in I$ and $z \in \hat{Z}$, the conditional strict preference set $P^i(\hat{x}^i, \hat{z}; z)$ must be a non-empty subset of $X^i(z)$. But in this Section we go further and assume as well that each $P^i(\hat{x}^i, \hat{z}; z)$ is a subset of the interior of $X^i(z)$. Hence, the indifference curve which bounds this strict preference set is precluded from meeting the boundary of $X^i(z)$. Without this admittedly unsatisfactory additional interiority requirement, it is hard to see how to guarantee that any FSPCE is even weakly Pareto efficient.

These two extra assumptions enable the following natural counterpart of the usual first efficiency theorem of welfare economics to be proved:

THEOREM 1. *Any FSPCE is weakly Pareto efficient.*

PROOF: Let $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ be any feasible allocation. Suppose that the feasible allocation $(\mathbf{x}^I, \mathbf{y}^J, z)$ is strictly Pareto superior — i.e., that $(x^i, z)\ P^i\ (\hat{x}^i, \hat{z})$ for all $i \in I$. Because $A(z)$ is compact, it loses no generality to assume that $(\mathbf{x}^I, \mathbf{y}^J)$ is (constrained) Pareto efficient in the conditional economy given z. By the standard assumptions set out in Section 2.1, there must exist a price vector $p^* \neq 0$ at which the allocation $(\mathbf{x}^I, \mathbf{y}^J, z)$ is private good compensated competitive. Because $(x^i, z)\ P^i\ (\hat{x}^i, \hat{z})$, the extra assumption set out above implies that x^i is an interior point of $X^i(z)$. Because preferences are continuous, there must exist $\tilde{x}^i \in X^i(z)$ such that $p^*\, \tilde{x}^i < p^*\, x^i$ and $(\tilde{x}^i, z)\ P^i\ (\hat{x}^i, \hat{z})$. It follows that $p^*\, x^i > e^i(p^*, z)$ for all $i \in I$. Therefore

$$\sum\nolimits_{j \in J} \pi^j(p^*, z) = \sum\nolimits_{j \in J} p^*\, y^j = \sum\nolimits_{i \in I} p^*\, x^i > \sum\nolimits_{i \in I} e^i(p^*, z)$$

implying that $\hat{S}(p^*, z) > 0$. Now construct $\mathbf{w}^I(p)$ and $\mathbf{t}^I(p)$ to satisfy

$$w^i(p) - t^i(p) = \hat{e}^i(p, z) + \theta^i\, \hat{S}(p^*, z)$$

for all $i \in I$ and $p \neq 0$ where, in order to ensure that $w^i(p^*) - t^i(p^*) = p^*\, x^i$, one chooses

$$\theta^i := [p^*\, x^i - \hat{e}^i(p^*, z)]/\hat{S}(p^*, z) > 0$$

Then $\sum_{i \in I} \theta^i = 1$, of course. Also, construct $s^j(p) := \pi^j(\hat{p}, \hat{z}) - \pi^j(p, z)$ for all $j \in J$ and $p \neq 0$.

These constructions make $(z, \mathbf{w}^I(p), \mathbf{t}^I(p), \mathbf{s}^J(p))$ a generalized public sector proposal which violates part (ii) of the definition of an FSPCE in Section 3.1. This implies that $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ cannot be an FSPCE allocation at any price vector $\hat{p} \neq 0$.

On the other hand, when $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z}, \hat{p})$ is an FSPCE, it follows that the allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ must be Pareto efficient. ∎

3.4. Second Efficiency Theorem

The following counterpart of the usual second efficiency theorem is true without any additional assumptions:

THEOREM 2. *Suppose $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ is a weakly Pareto efficient allocation. Then there is a price vector $\hat{p} \neq 0$ such that $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z}, \hat{p})$ is a compensated FSPCE.*

PROOF: Let $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ be any feasible allocation. Suppose there is a generalized public sector proposal $(z, \mathbf{w}^I(p), \mathbf{t}^I(p), \mathbf{s}^J(p))$ with an associated allocation $(\mathbf{x}^I, \mathbf{y}^J)$ which is private good competitive at a price vector p^* satisfying $\hat{S}(p^*, z) > 0$. Then condition (d) ensures that the cheaper point Lemma of Sction 2.4 applies, and also that the allocation $(\mathbf{x}^I, \mathbf{y}^J, z)$ is both feasible and Pareto superior.

Conversely, if $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ is weakly Pareto efficient, there can be no such generalized public sector proposal. Moreover, by the usual second efficiency theorem of welfare economics, there must exist a price vector $\hat{p} \neq 0$ at which the Pareto efficient allocation is private good compensated competitive. The above definitions imply that $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z}, \hat{p})$ is a compensated FSPCE. ∎

4. Valuation Equilibrium

4.1. Self-Financing and Balanced Valuation Schemes

In the tradition of Lindahl's pioneering work, it will now be assumed that the environment z is determined by unanimous choice as a result of some pricing scheme. Like Mas-Colell (1980), this pricing scheme will typically be non-linear. But as in the recent work by Diamantaras *et al.*, we extend the pricing scheme to accommodate many private goods. In fact we go beyond their work by allowing the valuation scheme for the public environment z to depend on the price vector p for private goods, since agents may not know this in advance.

In fact, let the unit sphere $P := \{ p \in \Re^G \mid \sum_{g \in G} {p_g}^2 = 1 \}$ be the private good normalized price domain. Then each individual $i \in I$ will be required to pay a net amount $\tau^i(p, z)$, as a function of (p, z) defined on some domain $D \subset P \times Z$. In addition, each firm $j \in J$ will receive a net subsidy $\sigma^j(p, z)$, also defined on D. A *valuation scheme* is defined as a collection (τ^I, σ^J) consisting of one complete profile $\langle \tau^i(p, z) \rangle_{i \in I}$ of consumer tax or payment functions, together with a second complete profile $\langle \sigma^j(p, z) \rangle_{j \in J}$ of producer subsidy or revenue functions.

Say that the valuation scheme is *self-financing* if $\sum_{j \in J} \sigma^j(p, z) \leq \sum_{i \in I} \tau^i(p, z)$ for all $(p, z) \in D$. This simply requires the valuation scheme to earn a non-negative profit because aggregate net subsidies paid to producers do not exceed aggregate net payments by consumers. On the other hand, the valuation scheme is *balanced at* $(p, z) \in D$ if $\sum_{i \in I} \tau^i(p, z) = \sum_{j \in J} \sigma^j(p, z)$. Thus, balance requires aggregate net payments by consumers to equal aggregate net subsidies to firms — as they do in cost-sharing mechanisms, for example. In particular, the degenerate valuation scheme satisfying $\sigma^j(p, z) = \tau^i(p, z) = 0$

for all i, j and all $(p, z) \in P \times Z$ is balanced everywhere. Trivially, any scheme that is balanced everywhere must be self-financing, though the reverse is obviously not true.

4.2. Valuation Equilibria

The following definition is reminiscent of the generalization of Lindahl equilibrium due to Mas-Colell, Diamantaras, and others. It also extends to many private goods Kaneko's (1977) concept of ratio equilibrium — see also Mas-Colell and Silvestre (1989, 1991). Relative to a valuation scheme $\tau^I(p, z)$, $\sigma^J(p, z)$, a *valuation equilibrium with lump-sum transfers* (or VELT) is a collection $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z}, \hat{p})$ of conditionally feasible individual plans $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J) \in \mathbf{X}^I(\hat{z}) \times \mathbf{Y}^J(\hat{z})$ for consumers and producers, together with a public environment $\hat{z}$ and a private good price vector $\hat{p}$, such that:

(i) the valuation scheme is defined and balanced at $(\hat{p}, \hat{z})$;

(ii) $\sum_{i \in I} \hat{x}^i = \sum_{j \in J} \hat{y}^j$;

(iii) for all $j \in J$ and $(p, z) \in D$, if $y^j \in Y^j(z)$ then $p\, y^j + \sigma^j(p, z) \leq \hat{p}\, \hat{y}^j + \sigma^j(\hat{p}, \hat{z})$;

(iv) for all $i \in I$ and $(p, z) \in D$, if $(x^i, z)\ P^i\ (\hat{x}^i, \hat{z})$ then $p\, x^i + \tau^i(p, z) > \hat{p}\, \hat{x}^i + \tau^i(\hat{p}, \hat{z})$.

Note that many producers have replaced the single producer who appears in the work of Mas-Colell, Diamantaras, *et al.* Like consumers, producers are also faced with non-linear Lindahl prices. In valuation equilibrium, no agent can deviate to a better alternative allocation. As in the usual formulation of Lindahl equilibrium, the valuation scheme provides incentives that result in the equilibrium public environment $\hat{z}$ being chosen unanimously.

In fact, producers could be entirely disregarded when seeing if there is a consensus. Instead, they could simply be commanded to do their part in bringing about $\hat{z}$. After all, the net subsidies which firms receive are merely passed on to consumers. However, we follow normal first-best theory in requiring a firm to receive compensation for any public goods it produces, and also if its set-up costs are too large to be covered when prices are set equal to marginal cost.

A *compensated valuation equilibrium* is defined similarly, the only difference being that condition (iv) is replaced by:

(iv′) for all $i \in I$ and $(p, z) \in D$, if $(x^i, z)\ R^i\ (\hat{x}^i, \hat{z})$ then $p\, x^i + \tau^i(p, z) \geq \hat{p}\, \hat{x}^i + \tau^i(\hat{p}, \hat{z})$.

The cheaper point lemma of Section 2.4 provides sufficient conditions for a compensated valuation equilibrium to be a valuation equilibrium.

4.3. Proper Valuation Schemes and the First Efficiency Theorem

Recall the definition of $\hat{Z}$ as the subset of Z consisting of those z for which each consumer i's conditional weak preference set $R^i\,(\hat{x}^i, \hat{z}; z)$ is non-empty. Say that the valuation scheme $\tau^I(p,z)$, $\sigma^J(p,z)$ is *proper* provided that, for each $z \in \hat{Z}$, there exists at least one $p \in P$ such that $(p,z) \in D$ and the scheme is balanced at (p,z).

Diamantaras and Gilles (1986) consider a valuation scheme which is both defined and balanced on a domain $D \subset P \times Z$ satisfying the property that, for each $z \in Z$, there exists a single price vector $p(z)$ for which $(p(z), z) \in D$. They suggest that $p(z)$ be interpreted as a common conjecture concerning what price vector will emerge from a change in the public environment. A special case would be if agents were far-sighted in the sense of Section 3. Clearly, any such valuation scheme is proper according to the definition we have just given. But our definition is more general, allowing agents to contemplate multiple possible price vectors in each public environment.

Like the usual first fundamental efficiency theorem of welfare economics, the first main result says that a valuation equilibrium is Pareto efficient. Generally, however, it is only valid if the valuation scheme is proper.

THEOREM 3. *If $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z}, \hat{p})$ is a VELT relative to a proper valuation scheme $\tau^I(p,z)$, $\sigma^J(p,z)$, then the allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ is Pareto efficient.*

PROOF: Suppose that $(x^i, z)\ R^i\ (\hat{x}^i, \hat{z})$ for all $i \in I$, with $(x^h, z)\ P^h\ (\hat{x}^h, \hat{z})$ for some $h \in I$. Suppose too that $y^j \in Y^j(z)$ for all $j \in J$. Because $z \in \hat{Z}$, there must exist $p \in P$ such that the proper valuation scheme is defined and balanced at (p,z). Then the above definitions and local non-satiation of consumers' preferences together imply that $\hat{p}\,\hat{y}^j + \sigma^j(\hat{p}, \hat{z}) \geq p\,y^j + \sigma^j(p,z)$ for all $j \in J$, and that $p\,x^i + \tau^i(p,z) \geq \hat{p}\,\hat{x}^i + \tau^i(\hat{p}, \hat{z})$ for all $i \in I$, with strict inequality when $h = i$. It follows that

$$\begin{aligned} \textstyle\sum_{i\in I} [p\,x^i + \tau^i(p,z)] &> \textstyle\sum_{i\in I} [\hat{p}\,\hat{x}^i + \tau^i(\hat{p},\hat{z})] \\ = \textstyle\sum_{j\in J} [\hat{p}\,\hat{y}^j + \sigma^j(\hat{p},\hat{z})] &\geq \textstyle\sum_{j\in J} [p\,y^j + \sigma^j(p,z)] \end{aligned}$$

Hence

$$p\left(\textstyle\sum_{i\in I} x^i - \sum_{j\in J} y^j\right) > \textstyle\sum_{j\in J} \sigma^j(p,z) - \sum_{i\in I} \tau^i(p,z) = 0$$

where the last equality follows because the valuation scheme is balanced at (p,z). It follows that there can be no Pareto superior allocation $(\mathbf{x}^I, \mathbf{y}^J, z)$ satisfying the feasibility constraint $\sum_{i\in I} x^i = \sum_{j\in J} y^j$. ∎

Observe that a valuation equilibrium allocation could be inefficient if there were some $z \in \hat{Z}$ such that for all $p \in P$ the valuation scheme were undefined or else lacked balance. For example, one or more firms could be encouraged to incur unnecessary extra costs through subsidies that need to be financed by unnecessarily large charges on consumers.

4.4. Regular Valuation Schemes and the Second Efficiency Theorem

The second result corresponds to the second fundamental efficiency theorem by establishing that any Pareto efficient allocation can be decentralized as a compensated valuation equilibrium relative to a suitable proper valuation scheme. But in fact a stronger result is possible, because the valuation scheme can be made to satisfy a stricter "regularity" condition.

Recall the definition of $\hat{S}(p,z) := \sum_{j \in J} \pi^j(p,z) - \sum_{i \in I} \hat{e}^i(p,z)$ as the surplus of aggregate maximum profit at (p,z) over the minimum total expenditure needed to make consumers no worse off than in the *status quo*, and the use of the inequality $\hat{S}(p,z) > 0$ as a cost–benefit test indicating that changing the public environment to z would be a potential Pareto improvement if p could emerge as an equilibrium price vector. Say that the valuation scheme $\tau^I(p,z)$, $\sigma^J(p,z)$ is *regular* if it is self-financing and, for each $z \in \hat{Z}$, there exists at least one $p \in P$ satisfying the following two properties simultaneously:

(i) $(p,z) \in D$ and the valuation scheme is balanced at (p,z);

(ii) either $\hat{S}(p,z) \leq 0$ or alternatively, if $\hat{S}(p,z) > 0$, then in the conditional private good economy given z there must be a feasible allocation $(\mathbf{x}^I, \mathbf{y}^J)$ which is competitive at a price vector $p \neq 0$ satisfying $p\, x^i > \hat{e}^i(p,z)$ for all $i \in I$.

Because of (i), a regular valuation scheme is proper. But it must also be self-financing. Then the extra requirement (ii) imposes a significant further strengthening, especially when z passes the test $\hat{S}(p,z) > 0$ for all $p \in P$. In this case, given z, there must be balance for at least one price vector that could emerge from a private good Walrasian equilibrium relative to a distribution rule specifying the wealth $w^i(p)$ of each consumer $i \in I$ as a function of p satisfying $\sum_{i \in I} w^i(p) = \sum_{j \in J} \pi^j(p,z)$ and also $w^i(p) > \hat{e}^i(p,z)$ for all $p \in P$. Because preferences are locally non-satiated, the allocation $(\mathbf{x}^I, \mathbf{y}^J)$ resulting from any such equilibrium evidently satisfies $(x^i, z)\ P^i\ (\hat{x}^i, \hat{z})$ for all $i \in I$. In fact, when $\hat{S}(p,z) > 0$, property (ii) is similar to the test used in Section 3.2 when defining a far-sighted public competitive equilibrium.

THEOREM 4. *Let* $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ *be a weakly Pareto efficient allocation. Then, under the assumptions of Section 2.1, there exists a price vector* $\hat{p} \in P$ *and a regular valuation scheme* $\tau^I(p,z)$, $\sigma^J(p,z)$ *defined on the whole domain* $D = P \times Z$ *relative to which the allocation is a compensated VELT at the price vector* $\hat{p}$.

PROOF: For each $z \in \hat{Z}$, define

$$B^i(z) := \{\, x^i \in X^i(z) \mid (x^i, z)\ P^i\ (\hat{x}^i, \hat{z}) \,\}$$

as the (non-empty) set of net trade vectors allowing i to be better off than at $(\hat{x}^i, \hat{z})$, but in the public environment z instead of $\hat{z}$. Because $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ is weakly Pareto efficient, the two non-empty convex sets $\sum_{i\in I} B^i(z)$ and $\sum_{j\in J} Y^j(z)$ are disjoint. So there is a non-empty set $\Pi(z)$ of price vectors $p \in P$ which each determine a separating hyperplane $p\,x = \alpha$ such that, whenever $x \in \sum_{i\in I} B^i(z)$ and $y \in \sum_{j\in J} Y^j(z)$, then $p\,x \geq \alpha \geq p\,y$. In particular, because there exists x^i in the closure of each $B^i(z)$ such that $p\,x^i = \hat{e}^i(p,z)$, and also there exists y^j in each Y^j such that $p\,y^j = \pi^j(p,z)$, it follows that

$$\sum\nolimits_{i\in I} \hat{e}^i(p,z) \geq \alpha \geq \sum\nolimits_{j\in J} \pi^j(p,z) \tag{$*$}$$

This implies that $\hat{S}(p,z) \leq 0$ for all $p \in \Pi(z)$.

Because $\hat{z} \in \hat{Z}$, one can choose $\hat{p} \in P$ as any member of the non-empty set $\Pi(\hat{z})$. But $\hat{p}\,\hat{x}^i \geq \hat{e}^i(\hat{p},\hat{z})$ for all $i \in I$ and $\hat{p}\,\hat{y}^j \leq \pi^j(\hat{p},\hat{z})$ for all $j \in J$. Also, because $(*)$ applies when $(p,z) = (\hat{p},\hat{z})$, it follows that

$$\sum\nolimits_{i\in I} \hat{p}\,\hat{x}^i \geq \sum\nolimits_{i\in I} \hat{e}^i(\hat{p},\hat{z}) \geq \alpha \geq \sum\nolimits_{j\in J} \pi^j(\hat{p},\hat{z}) \geq \sum\nolimits_{j\in J} \hat{p}\,\hat{y}^j$$

Now, because of local non-satiation, the aggregate net trade vector $\sum_{i\in I} \hat{x}^i = \sum_{j\in J} \hat{y}^j$ belongs to the intersection of $\sum_{j\in J} Y^j(\hat{z})$ with the closure of $\sum_{i\in I} B^i(\hat{z})$. So the hyperplane $\hat{p}\,x = \alpha$ must pass through this point of the intersection, implying that $\sum_{i\in I} \hat{p}\,\hat{x}^i = \sum_{j\in J} \hat{p}\,\hat{y}^j = \alpha$. It follows that $\hat{p}\,\hat{x}^i = \hat{e}^i(\hat{p},\hat{z})$ for each $i \in I$ and that $\hat{p}\,\hat{y}^j = \pi^j(\hat{p},\hat{z})$ for each $j \in J$.

Next, let $\hat{\sigma}^J$ and $\hat{\tau}^I$ be profiles of arbitrary constants satisfying $\sum_{j\in J} \hat{\sigma}^j = \sum_{i\in I} \hat{\tau}^i$. Let $\langle \alpha^j \rangle_{j\in J}$ be any profile of positive marginal profit shares that are paid as subsidies to firms, with $\sum_{j\in J} \alpha^j = 1$. Consider the valuation scheme defined on the whole of $P \times Z$ by

$$\sigma^j(p,z) := \hat{p}\,\hat{y}^j + \hat{\sigma}^j - \pi^j(p,z) + \alpha^j \min\{\, 0, \hat{S}(p,z) \,\} \quad (\text{all } j \in J)$$

$$\text{and} \quad \tau^i(p,z) := \hat{p}\,\hat{x}^i + \hat{\tau}^i - \hat{e}^i(p,z) \quad (\text{all } i \in I)$$

Then the conclusion of the previous paragraph implies that $\tau^i(\hat{p}, \hat{z}) = \hat{\tau}^i$ for each $i \in I$ and that $\sigma^j(\hat{p}, \hat{z}) = \hat{\sigma}^j$ for each $j \in J$.

Now, feasibility of the allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{z})$ implies that $\sum_{i \in I} \hat{x}^i = \sum_{j \in J} \hat{y}^j$. Also, we assumed that $\sum_{i \in I} \hat{\tau}^i = \sum_{j \in J} \hat{\sigma}^j$. Therefore,

$$\begin{aligned} \sum_{i \in I} \tau^i(p, z) - \sum_{j \in J} \sigma^j(p, z) &= \sum_{j \in J} [\pi^j(p, z) - \alpha^j \min\{0, \hat{S}(p, z)] - \sum_{i \in I} \hat{e}^i(p, z) \\ &= \hat{S}(p, z) - \min\{\hat{S}(p, z), 0\} = \max\{0, \hat{S}(p, z)\} \geq 0 \end{aligned}$$

with equality if and only if $\hat{S}(p, z) \leq 0$. Hence, the valuation scheme is always self-financing, and is balanced whenever $\hat{S}(p, z) \leq 0$. Because $\Pi(z)$ is non-empty for each $z \in \hat{Z}$ and $\hat{S}(p, z) \leq 0$ for all $p \in \Pi(z)$, it follows that this valuation scheme is regular.

Our construction implies that for all $j \in J$ and $(p, z) \in D$, whenever $y^j \in Y^j(z)$ then

$$p\, y^j + \sigma^j(p, z) \leq \pi^j(p, z) + \sigma^j(p, z) \leq \hat{p}\, \hat{y}^j + \hat{\sigma}^j = \hat{p}\, \hat{y}^j + \sigma^j(\hat{p}, \hat{z})$$

Similarly, for all $i \in I$ and $(p, z) \in D$, whenever $(x^i, z)\ R^i\ (\hat{x}^i, \hat{z})$ then

$$p\, x^i + \tau^i(p, z) \geq \hat{e}^i(p, z) + \tau^i(p, z) = \hat{p}\, \hat{x}^i + \hat{\tau}^i = \hat{p}\, \hat{x}^i + \tau^i(\hat{p}, \hat{z})$$

It has been verified that all four parts (i), (ii), (iii), and (iv′) of the above definition of a compensated VELT are satisfied. ▌

In proving Theorem 4, the valuation scheme has been given an explicit form with an easy and sensible interpretation. Each $\sigma^j(p, z)$ consists of four terms, of which the first three make the change in net subsidy exactly offset the firm's net decrease in profits. As for the last term that is always non-positive, whenever the net benefit $\hat{S}(p, z)$ is negative, it represents firm j's share of the total contribution $-\hat{S}(p, z)$ needed to finance this negative amount; otherwise the last term is zero. On the other hand, the extra payment $\tau^i(p, z) - \hat{\tau}^i$ demanded from consumer i is equal to the Hicksian compensating variation associated with the change from $(\hat{p}, \hat{z})$ to (p, z).

5. Concluding Remarks

5.1. Assessment

The Wicksell–Foley approach to determining the public environment involves a notion of politico-economic equilibrium where there is a negative consensus in the sense that no alternative public proposal is unanimously preferred. Section 3 set out to demonstrate the two efficiency theorems using this notion of equilibrium. When the public environment involves non-convexities, it seems that agents must foresee the pecuniary externalities that arise because changing the public environment alters prices for private goods. This permits acceptable results, though the first theorem of the text relies on a somewhat restrictive assumption set out in Section 3.3.

The usual efficiency theorems of welfare economics concern Walrasian equilibria in economies with only private goods, or Lindahl equilibria in economies with both public and private goods, or "Lindahl–Pigou" equilibria in economies with externalities as well as both public and private goods — see Hammond (1998) for a recent exposition. When the public environment involves non-convexities, Mas-Colell (1980) introduced the concept of a valuation equilibrium for economies with only one private good. A valuation equilibrium is like a Lindahl equilibrium, but with non-linear pricing of the public environment. In this special framework, he was able to derive convincing versions of the usual efficiency theorems of welfare economics, as well as a version of core equivalence.

Section 4 turned to this Lindahl approach, involving a positive consensus regarding what public environment should be chosen. Mas-Colell's versions of the efficiency theorems were generalized for an economy with many private goods, with assumptions that are somewhat less restrictive than those invoked by Diamantaras *et al.* Our results require a "proper" valuation scheme that is financially balanced on a suitable regular domain whose members are pairs consisting of a price vector together with a public environment. If agents ignore the dependence of prices on the public environment, however, generally it is impossible to have a proper valuation scheme. Hence, our results require agents to recognize how private good prices may depend on the public environment. Of course, when there is only one private good, this dependence does not matter and our definition reduces to that of Mas-Colell.

5.2. Adjustment to Equilibrium

The classical Walrasian theory of general competitive equilibrium invokes an auctioneer whose task it is to steer the economy toward an equilibrium price vector through a *tâtonnement* process. Because a Lindahl equilibrium is merely a Walrasian equilibrium for an economy in which there is a separate copy of the public environment for each individual agent, the Walrasian auctioneer could also be used to reach a Lindahl equilibrium.

Nevertheless, it is more intuitive to regard the public environment, together with the means of financing it, as emerging from a political process. Indeed, this view is made quite explicit in the Wicksell–Foley definition of a public competitive equilibrium. Perhaps with this in mind, Malinvaud (1971, 1972) together with Drèze and de la Vallée Poussin (1971) suggested what generally came to be known as the MDP procedure. In Malinvaud's approach, prices adjust to excess demands as in a Walrasian *tâtonnement*, while quantities of public goods adjust to the difference between total marginal willingness to pay and the marginal cost of producing those public goods. On the other hand, Drèze and de la Vallée Poussin (1971) adjust quantities even of private goods.

For the case when public goods are subject to non-convexities and there is only one private good, Mas-Colell (1980) proposed a "global version" of the MDP procedure. Of course, when there is only one private good, there is no need for the *tâtonnement* part of the MDP procedure. In future work, it remains to be seen whether when there are many private goods, one can re-introduce the *tâtonnement* part of the MDP procedure in order to reach a Pareto efficient allocation, as Malinvaud (1972) does for convex economies. Or whether a quantity guided procedure is needed even for private goods. Or whether, as suggested by the results in Section 3 especially, it will be necessary to allow the procedure first to converge to some kind of equilibrium in the private good economy given the public environment z, before applying a cost–benefit test which decides whether it is worth moving to z.

5.3. Incentive Constraints and the Second Best

A major limitation of our results so far is the neglect of private information. This is especially acute when one considers the public environment for which, as is well known, individuals typically have incentives to misreport their true willingness to pay. Of course, similar concerns apply to the MDP procedure itself, as many have recognized. It is all but impossible to avoid manipulation by agents who focus on the allocation to which the procedure eventually converges, rather than myopically on the direction of movement.

Nevertheless, the results reported in Hammond (1979, 1987) on mechanisms for economies with public goods and a continuum of agents may still be helpful. Under appropriate smoothness assumptions concerning private goods, these results show that the only hope of reaching a first-best Pareto efficient allocation using a strategy proof mechanism is to make the market value of the private goods allocated to each consumer independent of their preferences for the public environment, or of any other private information. In particular this suggests that public goods have to be financed by "poll taxes" which are levied regardless of individual circumstances.

Of course, even poll taxes are manipulable if some agents can plausibly plead an inability to pay them. Private information, therefore, is likely to lead in general to binding incentive constraints which make it impossible to achieve any first-best Pareto efficient allocation at all. Public goods, the public environment, and firms' set-up costs all have to be financed by distortionary taxes or by prices that exceed marginal cost. These are distortions which, by the way, can only arise in an economy with many private goods.[4]

However, in the case of production with set-up costs or other non-convexities, this view may be unduly pessimistic. Often these non-convexities apply to the production of intermediate goods, like aircraft which are bought only by airlines, or microprocessors which are bought only by computer manufacturers. Then, instead of consumers' demands for public goods, it is other producers' derived demands that are relevant. To the extent that these depend on observable technology, such demands may well be less subject to manipulation.

In any case, this is another issue which has to be left for later work. Our main conclusion remains — namely, the need to think of any "lumpy" decision in the economy as likely to

[4] We owe this important point to Agnar Sandmo.

give rise to pecuniary externalities. These imbue such decisions with many of the essential features of decisions affecting public goods or the public environment.

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