Asymptotically Strategy-Proof Walrasian Exchange

José M. Córdoba

ANESCO, Tonalá 42, Colonia Roma, 06700 México D.F., México

and Peter J. Hammond

Department of Economics, Stanford University, CA 94305-6072, U.S.A.

Address for editorial correspondence:

Prof. Peter J. Hammond

Department of Economics

Stanford University

Stanford, CA 94305–6072

U.S.A.

e-mail: hammond@leland.stanford.edu; fax: +1 (650) 725-5702

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Abstract

In smooth exchange economies with a continuum of agents, any Walrasian mechanism is Pareto efficient, individually rational, anonymous, and strategy-proof. Barberà and Jackson's recent results imply that no such efficient mechanism is the limit of resource-balanced, individually rational, anonymous and non-bossy strategy-proof allocation mechanisms for an expanding sequence of finite economies. For a broad class of smooth random exchange economies, relaxing anonymity and non-bossiness admits mechanisms which, as the economy becomes infinitely large, are asymptotically Walrasian for all except one "balancing" agent, while being manipulable with generically vanishing probability. Also considered are some extensions to non-Walrasian mechanisms.

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Asymptotic Strategy-Proofness

1. Introduction and Outline

1.1. Strategy-Proofness in Continuum Economies

In the literature on incentive compatibility, an important class of economic allocation mechanisms are those satisfying strategy-proofness — i.e., for each profile of individual characteristics or types, the outcome is the same as that of an equivalent direct mechanism in which each agent has truthful revelation as a dominant strategy. For general economic environments, there has been much past work on (usually symmetric or anonymous) strategy-proof mechanisms in continuum economies having a non-atomic measure space of agents. Such continuum economies can be viewed as limits of finite economies, with a large but finite number of agents. Accordingly, it would be desirable to find strategy-proof mechanisms for finite economies that converge to strategy-proof mechanisms for continuum economies as the number of agents tends to infinity. So far, no such limit theorem has been provided for general economic environments.

In general equilibrium theory, continuum economies were first considered by Aumann [2, 3]. His path-breaking articles led to a body of significant work that is well summarized in Hildenbrand [18]. Early work that pays attention to incentive issues in continuum economies can be found in the Vickrey [35] and Mirrlees [27] discussions of optimal redistributive income taxation. See also Rothschild and Stiglitz [32] and many other works embodying self-selection constraints. As argued in [15, 16, 17], an important motivation for studying incentive constraints in continuum economies is to understand better some of the limitations on economic policy imposed by private information. This is especially true in connection with financing public goods, with redistributive welfare programs, and with ensuring that free trade or other market liberalization policies lead to actual Pareto improvements when gainers are required to over-compensate those who would otherwise lose.

One good reason for considering continuum economies is that, unlike in finite economies, each individual's attempt to manipulate the economic environment has no effect on the apparent statistical distribution of relevant personal characteristics in the economy. This greatly simplifies the incentive constraints because the allocation that is chosen for one distribution of characteristics puts no restriction on what can be chosen for other distributions. Such independence between different continuum economies allows an abundance of strategy-proof mechanisms. It also permits relatively simple decentralization results of the kind presented in [17] — see also [8, 9, 25]. First-best Pareto efficiency is achievable and, in particular, any Walrasian mechanism is strategy-proof, anonymous, and individually rational.

In fact incentive constraints in continuum economies become identical to the often studied self-selection constraints that would apply if the distribution of characteristics were commonly known. In that case, truthful revelation turns out to be a dominant strategy provided the mechanism designer imposes a high enough penalty whenever the reported distribution of characteristics differs from the true one. However, when there is private information, it is hard to believe that the distribution of agents' characteristics could be known to the mechanism designer. For this reason, self-selection constraints acquire much of their interest only because they happen to correspond to incentive constraints in continuum economies. In this paper, we do not assume that the mechanism designer knows the distribution of characteristics which prevails in any finite economy; nor do we presume any knowledge of the limiting distribution of characteristics. The mechanisms we construct will depend only on agents' revelations concerning their unknown profile of characteristics. In this sense, our mechanisms will be "non-parametric", as defined by Hurwicz [20, p. 310]. This makes them more widely applicable than the "parametric" mechanisms considered by Dierker and Haller [11] and by Mas-Colell and Vives [26], among others, which rely on prior information about the distribution of characteristics.

1.2. Strategy-Proofness in Large Finite Economies

Despite their frequent use, the interpretation and practical relevance of continuum economy mechanisms has long remained unsatisfactory. There are still no good counterparts for finite economies. This is in marked contrast to results concerning the core of an exchange economy, for instance — see especially the surveys by Hildenbrand [19], Mas-Colell [24, Section 7.4], and Anderson [1]. The core is non-empty in economies with a finite number of agents for which a Walrasian equilibrium exists. Moreover, the core often converges as a "competitive sequence" of economies tends to a continuum economy, with limit equal to the set of Walrasian equilibrium allocations.

With few exceptions, limit theorems for strategy-proof mechanisms have been proved only for Vickrey–Clarke–Groves mechanisms that apply to special public good environments where agents are assumed to have quasi-linear preferences. For these mechanisms, Green and Laffont [14] proved that, when the project space is finite, the expected per capita tax and the expected total taxes collected in a Vickrey–Clarke–Groves mechanism both converge to zero as the number of participants goes to infinity. This result was generalized by Rob [30] to permit more general probability distributions on the sample space of participants' project valuations, and by Mitsui [28] for the case of an infinite project space. Since the size of the surplus is a direct measure of the inefficiency loss, these mechanisms are asymptotically efficient. But for strategy-proof mechanisms applying to more general economic environments such as exchange economies, no such general limit theorem is yet available.

One reason for this lack may be that it seems difficult to construct any satisfactory strategy-proof mechanism at all for a general exchange economy with a finite set of agents. In this connection, an important recent result is due to Barberà and Jackson [5]. They prove that, on the domain of utility functions that are continuous, strictly quasi-concave, and increasing, an allocation rule for an exchange economy is resource-balanced, strategy-proof, individually rational, anonymous, and non-bossy if and only if it is the result of anonymous fixed-proportion trading. This result suggests that, for general finite exchange economies, strategy-proofness entails a significant loss of efficiency. As stated in [5, p. 65]:

> "...the intuition is roughly as follows. To satisfy strategy-proofness, a rule cannot choose the direction of trade based on the size of desired trades in various directions. Such a rule would be vulnerable to manipulability due to potential gains from overstating or understating the desire to trade. Strategy-proofness thus restricts the choice of the direction to trade to depend only on the signs of desired trades in various directions."

As these authors also remark [5, p. 60]:

> "Such rules are not efficient since the selection of the trading price is not responsive to the exact demands of the agents, but only to whether they are positive or negative."

In fact the conditions in [5] imply that trade can occur along only a finite collection of predefined straight lines whose maximum number is proportional to the number of agents. As the number of agents tends to infinity, so does the maximum number of lines. But so also does the dimension of the generalized Edgeworth–Bowley box describing the set of feasible allocations. Hence, even in the limit these lines form a nowhere dense set of measure zero in the commodity space. So generically an efficient allocation of an exchange economy will not be close to such a set. As concluded in [5, p. 66]: "These . . . restrictions on strategy-proof rules make it clear that there is no hope for any sort of approximate efficiency result in the limit (as the economy grows)."

In particular, their result seems to suggest that a Pareto-efficient mechanism (for instance, the Walrasian mechanism) in a continuum economy cannot be the limit of any sequence of strategy-proof mechanisms for an expanding sequence of finite economies having the continuum economy as a limit. Such a conclusion calls into question both the practical relevance of existing work on strategy-proofness in continuum economies and the scope of standard asymptotic results on incentive compatibility for expanding economies.

1.3. A Conjecture

Ideally, for a general profile of characteristics and for the general domain of increasing, continuous and strictly convex preferences, one would like to construct a convergent sequence of strategy-proof allocation mechanisms which are individually rational and resource-balanced for each large economy in a converging sequence. Moreover, one would like the limit allocation mechanism to be efficient in the continuum limit economy. Obviously, we know from [5] that the assumptions of anonymity and/or non-bossiness would have to be violated.

We conjecture that this ideal is unattainable because, over the general domain of increasing, strictly convex and continuous preferences, no mechanism is simultaneously individually rational, resource-balanced, strategy-proof and asymptotically Walrasian. Our intuition is that the uni-dimensionality of the range of the mechanisms analysed in [5] is not implied by non-bossiness or even by anonymity: rather, our conjecture is that no individually rational and strategy-proof rationing mechanism for unrestricted preferences has a range that includes, even in its closure, any connected component having dimension greater than one. If correct, this would make it impossible to reconcile strategy-proofness and asymptotic efficiency for general preference profiles.

1.4. Asymptotic Strategy-Proof Walrasian Mechanisms

Our mechanisms rely on separating out one balancing agent who, like Wilson's auctioneer in [37], is virtually passive. Wherever possible, this agent will be allocated a net trade vector that preserves balance, given what is allocated to the others. Then the other agents will be divided into two roughly equal groups. To try to ensure both individual rationality and strategy-proofness, each group is faced with a budget constraint that contains the no-trade option and otherwise depends only on the other group's reported types. In order to make individual rationality likely even for the balancing agent, other agents' demands are limited by imposing quadratic transaction fees. These fees and associated adjustments to the price vectors are intended to generate an aggregate net trade vector for the non-balancing agents whose components all converge to minus infinity as the economy becomes large, even as the net trade vector per head converges to zero.

We will define an economy as a probability distribution over agents' characteristics. We will consider increasing and nested finite sets of agents with characteristics drawn independently from this given distribution. We introduce a sequence of resource-balanced and individually rational mechanisms over this sequence of expanding economies and ensure, by construction, that the limiting mechanism is Walrasian.

Say that a mechanism is *locally strategy-proof* for a particular type profile if truthful revelation is a Nash equilibrium of the direct revelation game for that profile. It will turn out that, for a generic set of well-behaved limiting economies, our sequence of mechanisms makes the probability that the random type profile satisfies local strategy-proofness converge to 1 as the economy becomes infinitely large. Thus, in addition to satisfying resource-balance and individual rationality, the sequence of mechanisms is not only "asymptotically Walrasian", but is also "asymptotically strategy-proof" in an obvious sense.

In the rest of the paper, the preliminary Section 2 sets out notation, as well as the main definitions and assumptions. Section 3 describes the asymptotic Walrasian mechanisms to which our sequence of mechanisms converges almost surely as the number of agents tends to infinity. Then, for an expanding sequence of finite exchange economies, the key Section 4 constructs mechanisms that are resource-balanced, individually rational and asymptotically Walrasian. It is argued informally that, for a generic class of limit economies, these mechanisms are indeed asymptotically strategy-proof as the number of agents tends to infinity.

Section 5 proves formally that, under smoothness assumptions set out in Sections 3 and 4, this sequence of mechanisms has the claimed asymptotic properties. The final Section 6 discusses variations and possible extensions of the main result.

2. Preliminaries: Economies as Distributions

2.1. *Agents' Smooth Characteristics*

Let G denote the finite set of $\ell = \#G$ different *commodities*, so that $\Re^G$ is the (finite dimensional Euclidean) commodity space. Let $\Delta := \{ p \in \Re^G \mid \sum_{g \in G} p_g = 1 \}$ be the unit simplex of *normalized price vectors* in $\Re^G$, with relative interior $\Delta^0 := \Delta \cap \Re^G_{++}$ and with boundary bd $\Delta := \Delta \setminus \Delta^0$.

Let $I := \{ 0, 1, 2, \ldots \}$ be the countable set of possible agent names or *identifiers* i. Let Θ be the set of possible agent *characteristics* or *types* θ, each of which determines an ordered pair $(X(\theta), u(\cdot; \theta))$ made up of an *individually feasible set* $X(\theta)$ of net trade vectors $x \in \Re^G$, together with an ordinal equivalence class of *utility functions* $u(x; \theta)$ defined for all $x \in X(\theta)$.

Let Θ_S denote the set of all *smooth* agent characteristics, defined as those θ satisfying the following assumptions:

(i) the feasible set $X(\theta)$ is closed and convex, with a *lower bound* $\underline{x}(\theta)$ such that $x \in X(\theta)$ implies $x \geqq \underline{x}(\theta)$;

(ii) 0 is in the interior int $X(\theta)$ of $X(\theta)$;

(iii) $X(\theta)$ allows *free disposal* in the sense that, if $x \in X(\theta)$ and $x' \geqq x$, then $x' \in X(\theta)$;

(iv) there exists a utility function $u(x; \theta)$ of x which is C^2 for all $x \in X(\theta)$, with gradient vector satisfying $\partial u(x; \theta) \gg 0$, and which is also *differentiably strictly quasi-concave* in the sense that the Hessian quadratic form $v^\top \partial^2 u(x; \theta)\, v$ is negative definite for all $v \neq 0$ satisfying the constraint $\partial u(x; \theta)\, v = 0$;

(v) there exists $\underline{u}(\theta) \in \Re$ such that $u(x; \theta) \geq \underline{u}(\theta)$ for all $x \in X(\theta)$, with $u(x; \theta) = \underline{u}(\theta)$ iff $x \in$ bd $X(\theta)$.

Of these assumptions, only (v) is stronger than usual. Smoothness is generally taken to require that, whenever $\bar{x} \in$ int $X(\theta)$, then the upper-contour set

$$R(\bar{x}) := \{ x \in X(\theta) \mid u(x; \theta) \geq u(\bar{x}; \theta) \}$$

must remain within int $X(\theta)$. In this paper, assumption (v) is innocuous because it loses no generality to replace $X(\theta)$ with $R(\bar{x})$ for some $\bar{x} \in \text{int } X(\theta)$ with $\bar{x} \ll 0$.

2.2. Smooth Demands

Given any $\theta \in \Theta_S$, define the minimum feasible net wealth for all $p \in \Delta^0$ as

$$\underline{w}(p;\theta) := \min_x \{ p\,x \mid x \in X(\theta) \}$$

Evidently, the above conditions (i) and (ii) imply that $\underline{w}(p;\theta) < 0$ everywhere. For each $\theta \in \Theta_S$, consider the domain

$$D(\theta) := \{ (p,w) \in \Delta^0 \times \Re \mid w \geq \underline{w}(p;\theta) \}$$

of price–wealth pairs that are feasible for a θ-agent. Each price–wealth pair $(p,w) \in D(\theta)$ determines a *Walrasian budget set*

$$B^W(p,w) := \{ x \in \Re^G \mid p\,x \leq w \}$$

that is non-empty, convex, and compact. There is an associated single-valued *Walrasian net trade function* defined for $(p,w) \in D(\theta)$ by

$$(p,w) \mapsto x^W(p,w;\theta) := \arg\max_x \{ u(x;\theta) \mid x \in X(\theta) \cap B^W(p,w) \}$$

Conditions (i) to (v) ensure that for each $\theta \in \Theta_S$ all "points of demand are regular" in the sense of [24, Definition 2.7.1], even at the boundary of the feasible set $X(\theta)$. Hence, each $x^W(p,w;\theta)$ is a C^1 function of (p,w) throughout the domain $D(\theta)$. Also, each $\underline{w}(p;\theta)$ is a C^1 function of p throughout the domain Δ^0. Finally, note that $\|x^W(p,w;\theta)\| \to \infty$ whenever p converges to any $p^* \in \text{bd } \Delta$, where at least one good has a zero price.

By standard results in consumer demand theory, each $\theta \in \Theta_S$ corresponds to a unique ordinal equivalence class of *indirect utility functions* defined by

$$v(p,w;\theta) := u(x^W(p,w;\theta);\theta) = \max_x \{ u(x;\theta) \mid x \in X(\theta) \cap B^W(p,w) \}$$

which are homogeneous of degree zero throughout the domain of $(p,w) \in D(\theta)$. Note too that Roy's identity $\partial_p v(p,w;\theta) = -\lambda(p,w;\theta)\, x^W(p,w;\theta)$ holds everywhere with $\lambda(p,w;\theta) =$

$\partial_w v(p,w;\theta)$ and can be differentiated continuously w.r.t. p and w. This implies that $v(p,w;\theta)$ is a C^2 function of (p,w) which must also be differentiably strictly quasi-convex.

By the inverse function theorem, because $\partial_w v(p,w;\theta) > 0$, each $\theta \in \Theta_S$ also corresponds to a unique class of *expenditure functions* $E(p,u;\theta)$ satisfying $E(p,v(p,w;\theta);\theta) = w$ whenever $(p,w) \in D(\theta)$ and $u \geq \underline{u}(\theta)$. These functions are C^2 in (p,u), homogeneous of degree one in p, and have a negative semi-definite Hessian matrix $\partial_p^2 E(p,u;\theta)$ of rank $\ell - 1$, while satisfying $\partial_u E(p,u;\theta) > 0$ and also $E(p,\underline{u}(\theta);\theta) = \underline{w}(p;\theta)$.

2.3. *A Metric Space of Smooth Characteristics*

Now Θ_S must be given a suitable topology. There are two difficulties in doing so. One is that the domain $X(\theta)$ of each utility function varies with θ. On its own, this might be remedied using an approach similar to that of Back [4]. The second difficulty is that putting a typical metric on the space of utility functions seems unsatisfactory to us because it is bound to specify a positive distance between even ordinally equivalent pairs.[1] Instead, we modify an idea due to Kurt Hildenbrand and give a suitable family of demand functions a metric — see [8, Appendix to Ch. 2] as well as [10, p. 815].

First, for each $\theta \in \Theta_S$ and for the fixed domain $\Delta^0 \times \Re_+$ of pairs (p,y), independent of θ, define the function $g(\cdot,\cdot;\theta) : \Delta^0 \times \Re_+ \to \Re^G \times \Re$ by

$$g(p,y;\theta) := (x^W(p,\underline{w}(p;\theta) + y;\theta),\ \underline{w}(p;\theta))$$

This makes $g(p,y;\theta)$ a C^1 function of (p,y), whose derivative ∂g w.r.t. p and y is an $\ell \times (\ell+1)$ matrix. Moreover, as θ ranges over the space Θ_S of smooth characteristics satisfying (i) to (v) above, standard demand theory establishes a one-to-one correspondence between, on the one hand, the set of pairs $(X(\theta), u(\cdot;\theta))$, and on the other hand, the set of functions $g(p,y;\theta)$. Indeed, we have just shown how to construct $g(p,y;\theta)$ uniquely from $X(\theta)$ and $u(\cdot;\theta)$. Conversely, for any function $g(p,y;\theta)$ generated from $X(\theta)$ and $u(\cdot;\theta)$ in this way, one can easily reconstruct both the domain $D(\theta)$ and the function $x^W(p,w;\theta)$ defined on this domain; then the feasible set $X(\theta)$ must equal the closure of the range of $x^W(p,w;\theta)$,

[1] In fact, this difficulty might also be met by using the topology that Back [4] constructs for locally non-satiated preferences by establishing a homeomorphism with a subset of utility functions. However, it seems more natural to put a metric directly on demand functions.

whereas the utility function $u(x;\theta)$ is any suitable C^2 representation of the unique smooth preference ordering that is revealed by $x^W(p,w;\theta)$.

In this way, Θ_S is equivalent to a subset Γ_S of the space of C^1 functions $g : \Delta^0 \times \Re_+ \to \Re^G \times \Re$. But to give Θ_S a suitable metric, first we extend the range space $\Re^G \times \Re$ of each function $g \in \Gamma_S$ to its one-point compactification $(\Re^G \times \Re) \cup \{\infty\}$. Then we extend the domain of each function $g \in \Gamma_S$ from $\Delta^0 \times \Re_+$ to $\Delta \times \Re_+$ by defining $g^*(p,y) = \infty$ whenever $p \in \mathrm{bd}\ \Delta$. The resulting function $g^* : \Delta \times \Re_+ \to (\Re^G \times \Re) \cup \{\infty\}$ is continuous throughout its compact domain.

Next, recall that $\ell = \#G$. Then, for each $n = \ell, \ell+1, \ell+2, \ldots$, define the closed set

$$\Delta_n := \{\, p \in \Delta^0 \mid p_g \geq 1/n \quad (g \in G) \,\}$$

of price vectors with no component less than $1/n$. Then, using the Euclidean norm $\|\cdot\|$ for the spaces $\Re^G \times \Re$ and $\Re^{G\times G} \times \Re^G$ respectively, define the two pseudo-metrics d_n^0, d_n^1 on Θ_S by

$$d_n^0(\theta_1,\theta_2) := \sup_{p,y} \{\, \|g(p,y;\theta_1) - g(p,y;\theta_2)\| \mid (p,y) \in \Delta^0 \times [0,n] \,\}$$
$$d_n^1(\theta_1,\theta_2) := \sup_{p,y} \{\, \|\partial g(p,y;\theta_1) - \partial g(p,y;\theta_2)\| \mid (p,y) \in \Delta_n \times [0,n] \,\}$$

where d_n^0 is allowed to take the value $+\infty$. Finally, define the bounded metric d on Θ_S by

$$d(\theta_1,\theta_2) := \sum\nolimits_{n=\ell}^{\infty} 2^{-n} \left[\frac{d_n^0(\theta_1,\theta_2)}{1 + d_n^0(\theta_1,\theta_2)} + \frac{d_n^1(\theta_1,\theta_2)}{1 + d_n^1(\theta_1,\theta_2)} \right]$$

where $\frac{\infty}{1+\infty}$ should be regarded as equal to 1. This metric induces the topology of uniform C^1-convergence on compact subsets of the domain $\Delta^0 \times \Re_+$. As discussed by Mas-Colell [24, p. 50], this topology is separable and complete. So we can regard Θ_S as a complete separable metric space. In addition, when the product space $\Delta^0 \times \Re_+ \times \Theta_S$ is given its product topology, the mapping $(p,y;\theta) \mapsto (g(p,y;\theta), \partial g(p,y;\theta))$ must be jointly continuous in all three variables.

2.4. *Distributions of Characteristics*

In the limit as the number of agents becomes infinite, an economy will be regarded as a *distribution* of characteristics, taking the form of a probability measure defined on the Borel sets of Θ_S. We follow Mas-Colell [24, pp. 223–4] in restricting the domain of measures to have compact support, since we rely on some results that require such an assumption (for instance, continuity of mean demand and existence of Walrasian equilibrium, as well as local uniqueness of regular equilibria).

In this connection, it may be worth noting a sufficient condition for a given subset $K \subset \Theta_S$ to be compact. It is enough that K consist of C^2 functions of (p, y), while being both closed and (uniformly) bounded in the topology of uniform C^2-convergence on compact subsets of $\Delta^0 \times \Re_+$. This can be demonstrated using the equicontinuity argument in [33, p. 35].

So let $\mathcal{M}$ denote the set of all probability measures whose support is a compact subset of the metric space (Θ_S, d). The space $\mathcal{M}$ can be given the topology of weak convergence of measures, which corresponds to the Prohorov metric. Following Mas-Colell [24, p. 25], we endow $\mathcal{M}$ with a finer topology than that of weak convergence. In fact, we define a metric ρ on $\mathcal{M}$ so that, for all $\nu', \nu'' \in \mathcal{M}$, the distance $\rho(\nu', \nu'')$ is the sum of: (i) the Prohorov metric distance between ν' and ν''; (ii) the Hausdorff distance between the two supports supp ν' and supp ν''.

3. The Asymptotic Walrasian Mechanism

3.1. *Walrasian Excess Demands*

Given any distribution $\nu \in \mathcal{M}$ of agents' characteristics, let

$$z(p; \nu) := \int_{\Theta_S} x^W(p, 0; \theta)\, d\nu$$

denote the corresponding *Walrasian mean excess demand function*, with well–defined Jacobian

$$J(p) := \left(\frac{\partial z_g}{\partial p_h} \right)_{g,h \in G}$$

of partial derivatives w.r.t. prices. Because $z(p; \nu)$ must be homogeneous of degree 0 as a function of p, Euler's equation implies that $p\, J(p) = 0$ and so this matrix is singular.

However, we can ignore one good altogether, say good 1, since each agent's Walrasian demand must satisfy the budget constraint. Indeed, from now on let x_{-1} denote the vector $\langle x_g \rangle_{g \in G \setminus \{1\}}$. Also, re-normalize prices throughout so that commodity 1 is the *numéraire* with $p_1 = 1$. Thus, given equilibrium prices $p = (1, p_{-1})$, each agent's net trade vector $f(p;\theta) := x^W_{-1}(p, 0; \theta)$ will suffice to determine the Walrasian equilibrium allocation, with each agent's net trade of good 1 given by $x^W_1(p, 0; \theta) = -(1/p_1)\ p_{-1}\, f(p;\theta)$. Moreover, to describe the local comparative statics of the equilibrium price vector, it is enough to consider the $(\ell - 1) \times (\ell - 1)$ *reduced Jacobian matrix*

$$\partial z(p) := \left(\frac{\partial z_g}{\partial p_h} \right)_{g,h \in G \setminus \{1\}}$$

Because each $\nu \in \mathcal{M}$ has compact support, $\partial z(p)$ must be a continuous function of p on Δ^0.

3.2. The Open Set of Regular Distributions

Given any $\nu \in \mathcal{M}$, compactness of $\operatorname{supp} \nu$ guarantees the existence of a non-empty set $\Pi(\nu)$ of *Walrasian equilibrium* price vectors $p^W \in \Re^G_{++}$ satisfying $z(p^W; \nu) = 0$. In addition, by [18, Prop. 4, p. 152], the *Walrasian equilibrium price correspondence* $\nu \mapsto\!\!\!\rightarrow \Pi(\nu)$ has a closed graph.

Say that a particular Walrasian equilibrium price vector $p^W \in \Pi(\nu)$ is *regular* if $\partial z(p^W)$ is invertible, but that it is *critical* if $\partial z(p^W)$ is singular.

The distribution ν is said to be *regular* if every Walrasian equilibrium price vector $p^W \in \Pi(\nu)$ is regular; it is *singular* if some Walrasian equilibrium price vector is critical. Let $\mathcal{M}^*$ denote the set of regular distributions in $\mathcal{M}$. Because $\partial z(p)$ and its determinant are continuous on Δ^0, it follows as in [24, Prop. 5.8.14] that $\mathcal{M}^*$ is open relative to $\mathcal{M}$.

3.3. Regular Distributions Are Generic

Given any $\theta \in \Theta_S$ and $b \in \Re^G$ with $b \in \operatorname{int} X(\theta)$, there is a unique corresponding characteristic $\theta_b \in \Theta_S$ with feasible set $X(\theta_b) := X(\theta) - \{b\}$ and utility function $u(x;\theta_b) := u(x+b;\theta)$. In fact, the change from θ to θ_b is equivalent to giving a θ-agent the vector b as an extra endowment. The corresponding minimum wealth function satisfies $\underline{w}(p;\theta_b) = \underline{w}(p;\theta) - p\,b$, and the corresponding Walrasian demand function is given by $x^W(p, w; \theta_b) = x^W(p, w + p\,b; \theta) - b$. Hence $g(p, y; \theta_b) = g(p, y + p\,b; \theta) - b$. Routine calculations then confirm that $d(\theta_b, \theta) \to 0$ as $b \to 0$.

Let e denote the ℓ-dimensional vector $(1, 1, \ldots, 1) \in \Re^G$. Note that there exists a continuous function $\alpha : \Theta_S \to (0, 1]$ such that $-\alpha(\theta)\, e \in \operatorname{int} X(\theta)$ for all $\theta \in \Theta_S$ — for example, $\alpha(\theta) := \min \{ 1, \frac{1}{2} \max \{ \alpha \mid -\alpha\, e \in X(\theta) \} \}$. Given any $c \geqq -e$, let ν_c denote the distribution which is obtained from ν when each agent's characteristic shifts from θ to the corresponding θ_b, where $b = \alpha(\theta)c$. Thus, for every measurable set $K \subset \Theta_S$, one has $\nu_c(K) = \nu(\{ \theta \in \Theta_S \mid \theta_{\alpha(\theta)c} \in K \})$. Also, $\theta \in \operatorname{supp} \nu_c \iff \theta_{-\alpha(\theta)c} \in \operatorname{supp} \nu$. This implies that the Hausdorff distance between the supports of ν_c and ν converges to 0 as $c \to 0$. In addition, for every bounded continuous function $\phi : \Theta_S \to \Re$, one has

$$\int_{\Theta_S} \phi(\theta)\, d\nu_c = \int_{\Theta_S} \phi(\theta_{-\alpha(\theta)c})\, d\nu \to \int_{\Theta_S} \phi(\theta)\, d\nu$$

as $c \to 0$, because the support of any $\nu \in \mathcal{M}$ is compact. This shows that $\nu_c \to \nu$ in the topology we have given $\mathcal{M}$.

Next, note that the mean net trade function takes the form

$$z(p; \nu_c) = \int_{\Theta_S} x^W(p, p\, b; \theta)\, d\nu - \bar{\alpha}\, c, \quad \text{where} \quad \bar{\alpha} := \int_{\Theta_S} \alpha(\theta)\, d\nu > 0.$$

Consider the associated $\ell \times \ell$ matrix

$$\partial_c z(p; \nu_c) = \int_{\Theta_S} \partial_w x^W(p, p\, b; \theta)\, p^\top\, d\nu - \bar{\alpha}\, I$$

of derivatives w.r.t. the components of c. For every $a \in \Re^G$ satisfying $p\, a = 0$, it follows that the vector equation $\partial_c z(p; \nu_c)\, v = a$ has a solution $v = -(1/\bar{\alpha})\, a$. Also, $p\, \partial_c z(p; \nu_c) = 0$. Hence, there is a neighbourhood N of 0 in $\Re^G$ such that the matrix $\partial_c z(p; \nu_c)$ has rank $\ell - 1$ for all $p \in \Delta^0$ and all $c \in N$. So therefore does the $\ell \times 2\ell$ matrix $(\partial_p z(p; \nu_c), \partial_c z(p; \nu_c))$. From the transversality theorem, it follows as in [24, Prop. 5.8.16] that the distribution ν_c is regular for almost all c in some neighbourhood of 0. This proves that $\mathcal{M}^*$ is dense as well as open in $\mathcal{M}$.

3.4. A Walrasian Selection

A *selection* from the equilibrium correspondence is a mapping $\nu \mapsto p(\nu)$ satisfying $p(\nu) \in \Pi(\nu)$ for all $\nu \in \mathcal{M}$. So far there is nothing to guarantee the existence of a continuous selection that is defined on the whole domain $\mathcal{M}$, or even on the whole sub-domain $\mathcal{M}^*$ of regular distributions. However, adapting the proof of Mas-Colell [24, Prop. 5.8.18] to our topology establishes that a continuous selection does exist on some open and dense set $\mathcal{M}' \subset \mathcal{M}^*$. Moreover, because the set $\mathcal{M}$ is closed and the correspondence $\nu \mapsto\!\!\!\to \Pi(\nu)$ has a closed graph in $\mathcal{M} \times \Delta$, it follows from [18, Lemma 1, p. 55 and Prop. 1, p. 22] that there exists a selection which is measurable everywhere. This makes it possible to choose, with some degree of arbitrariness, a particular selection $\nu \mapsto p^W(\nu) \in \Pi(\nu)$ from the Walrasian equilibrium price correspondence which is measurable everywhere and continuous on an open and dense subset $\mathcal{M}'$ of $\mathcal{M}^*$, the set of regular distributions. This particular selection $p^W(\cdot)$ will be used to construct our asymptotically strategy-proof Walrasian mechanisms. To summarize, the following assumption is not vacuous because an appropriate selection does exist:

Assumption A.1. *The mapping $\nu \mapsto p^W(\nu)$ defined on $\mathcal{M}$ is a measurable selection from the correspondence $\nu \mapsto\!\!\!\to \Pi(\nu)$, and is continuous on some set $\mathcal{M}' \subset \mathcal{M}^*$ of regular distributions which is open and dense in $\mathcal{M}$.*

3.5. Weak Dispersion

Recall that our objective is to construct mechanisms for an expanding sequence of finite economies which generate allocations that converge to a Walrasian allocation in the limit. It is assumed that in each successive finite economy, agents' characteristics are drawn independently at random from an indefinitely large population with unknown true distribution $\bar{\nu} \in \mathcal{M}$. Clearly, if the selection $p^W(\cdot)$ happens to be discontinuous at or near $\bar{\nu}$, it will be virtually impossible to use this selection rule to determine a mechanism which converges to a Walrasian allocation in the limit economy with distribution $\bar{\nu}$. Accordingly, it will be assumed that $\bar{\nu}$ is in the open dense subset $\mathcal{M}'$, implying that $p^W(\cdot)$ is continuous in some neighbourhood N of $\bar{\nu}$.

From now on, let $\bar{p} := p^W(\bar{\nu})$ denote the value of the Walrasian price selection for the limit distribution $\bar{\nu}$.

Because $\bar{\nu} \in \mathcal{M}'$ and $\mathcal{M}'$ is a subset of the set of regular economies $\mathcal{M}^*$, at $\bar{p}$ the mean $(\ell - 1) \times (\ell - 1)$ reduced Jacobian matrix $\partial z(p)$ of the partial price derivatives of the mean excess demand must be invertible. We now impose one extra assumption to simplify later arguments by guaranteeing an invertible variance–covariance matrix for the net trade vectors and their price derivatives. The same assumption also guarantees that $S := \int_{\Theta_S} \|f(\theta)\|^2 d\bar{\nu} > 0$ which, as will be made clear below, ensures that the balancing agent can be paid a positive amount from the transaction costs which other agents are willing to pay. In the formal proofs this will enable proper estimates of the speed at which the Walrasian price sequence converges and of the probability that the mechanism fails to be locally strategy-proof.

To introduce the extra assumption, for each $h \in G \setminus \{1\}$, define $f'_h(\theta) := \frac{\partial f}{\partial p_h}(\theta) \in \Re^{G \setminus \{1\}}$. Then say that the distribution $\bar{\nu}$ is *weakly dispersed* if the ℓ different $(\ell-1) \times (\ell-1)$ symmetric matrices $\int_{\Theta_S} f(\theta) \, [f(\theta)]^\top d\bar{\nu}$ and $\int_{\Theta_S} f'_h(\theta) \, [f'_h(\theta)]^\top d\bar{\nu}$ $(h \in G \setminus \{1\})$ are all positive definite. This requires that, as θ varies over the support of the distribution $\bar{\nu}$, so each of the ℓ different vectors $f(\theta)$ and $f'_h(\theta)$ $(h \in G \setminus \{1\})$ in $\Re^{G \setminus \{1\}}$ must range over a set of full dimension $\ell - 1$. It also requires all the above matrices to be non-singular.

To summarize, we assume that:

Assumption A.2. *The unknown true distribution satisfies $\bar{\nu} \in \mathcal{M}'$ and is weakly dispersed.*

3.6. *Weakly Dispersed Distributions are Generic*

Because the relevant determinants are continuous functions of the distribution $\bar{\nu}$, the set of weakly dispersed distributions is open relative to $\mathcal{M}$. This subsection shows that the set of distributions satisfying a stronger dispersion condition is dense, so the set of weakly dispersed distributions is open and dense relative to $\mathcal{M}$.

First, given any symmetric $\ell \times \ell$ matrix A with all its elements non-negative and a zero diagonal, we follow an idea due to Diewert [12] in defining the *generalized Leontief price index*

$$P_A(p) := \sum_{g \in G} \sum_{h \in G \setminus \{g\}} a_{gh} \sqrt{p_g \, p_h}$$

Note that on the domain Δ^0 the function $P_A(p)$ is non-decreasing, concave, and homogeneous of degree 1.

Next, consider any non-negative symmetric matrix A as above, together with a vector $b \in \Re^G_+$ and a smooth characteristic $\theta \in \Theta_S$. As argued in Section 2.2, there is a unique corresponding smooth characteristic $\theta_{Ab} \in \Theta_S$ defined for all $p \in \Delta^0$ and all $u \geq \underline{u}(\theta) = \underline{u}(\theta_{Ab})$ by the expenditure function $E(p, u; \theta_{Ab}) := E(p, u; \theta) - p\, b + P_A(p)$ which is concave, homogeneous of degree 1, and C^2. The associated minimum expenditure functions are related by the equation $\underline{w}(p; \theta_{Ab}) := \underline{w}(p; \theta) - p\, b + P_A(p)$, and on the respective domains $D(\theta)$ and $D(\theta_{Ab})$, the corresponding indirect utility functions are related by the equation $v(p, w; \theta_{Ab}) := v(p, w + p\, b - P_A(p); \theta)$. Because $x^W(p, w; \theta) = \partial_p E(p, u; \theta)$ where $u = v(p, w; \theta)$, with a similar relation when θ is replaced by θ_{Ab}, it follows that

$$x^W(p, w; \theta_{Ab}) = x^W(p, w + p\, b - P_A(p); \theta) - b + \partial P_A(p)$$

The domain of characteristics $\theta_{Ab} \in \Theta_S$ which can be derived from any given θ in this way is homeomorphic to the non-negative orthant $\Re^m_+$ of a Euclidean space of dimension $m := \frac{1}{2}\ell(\ell + 1)$ because routine calculations establish that $d(\theta_{A_n b_n}, \theta_{Ab}) \to 0$ as $n \to \infty$ for the metric d given to Θ_S if and only if $\|(A_n - A, b_n - b)\| \to 0$ in the norm of $\Re^m$.

Now, given any fixed $p^* \gg 0$ and $b \in \Re^G$ satisfying $p^*\, b = 0$, choosing $A = 0$ ensures that $x^W(p^*, w; \theta_{Ab}) = x^W(p^*, w; \theta) - b$. Also, when $b = \partial_p P_A(p^*)$ and so $p^*\, b = P_A(p^*)$ because of Euler's theorem for homogeneous functions, it follows that

$$\partial_p x^W(p^*, w; \theta_{Ab}) = \partial_p x^W(p^*, w; \theta) + \partial^2 P_A(p^*)$$

Note that

$$\frac{\partial^2}{\partial p_g \partial p_h} P_A(p^*) = \tfrac{1}{2}\, a_{gh}/\sqrt{p_g\, p_h} \text{ if } g \neq h,$$

$$\text{whereas} \quad \frac{\partial^2}{\partial {p_g}^2} P_A(p^*) = -\tfrac{1}{2} \sum\nolimits_{h \neq g} a_{gh} \sqrt{p_h/{p_g}^3}.$$

Consider any non-zero vector $c = \langle c_g \rangle_{g \in G}$ with $p^*\, c = 0$ and $c_g \geq 0$ for each $g \neq h$, so $c_h < 0$. Then one can make $\frac{\partial}{\partial p_h} x^W_g(p^*, w; \theta_{Ab}) = \frac{\partial}{\partial p_h} x^W_g(p^*, w; \theta) + c_g$ for each $g \in G$ by choosing A and b so that

$$a_{gh} = 2\, c_g \sqrt{p^*_g\, p^*_h} \text{ for each } g \neq h,$$

$$\text{and} \quad b_g = \frac{\partial}{\partial p_g} P_A(p^*) = \sum\nolimits_{h \in G \setminus \{g\}} a_{gh} \sqrt{p^*_h/p^*_g} = 2\, c_g \sum\nolimits_{h \in G \setminus \{g\}} p^*_h \text{ for all } g \in G.$$

In effect, this merely confirms that $E(p, u; \theta_{Ab})$ really is a "flexible functional form" in the sense described by Diewert [12, p. 113].

Finally, given any $\theta \in \Theta_S$ and $\epsilon > 0$, define the compact set

$$\Theta_\epsilon(\theta) := \{ \theta_{Ab} \in \Theta_S \mid (A, b) \in \Re^m_+; \quad \|A\| + \|b\| \leq \epsilon \}$$

which has a non-empty interior in the homeomorphic subset $\Re^m_+$. Then, say that the distribution $\bar{\nu} \in \mathcal{M}$ is *dispersed* if there exist $\bar{\theta} \in \Theta_S$ and $\epsilon > 0$ such that $\Theta_\epsilon(\bar{\theta}) \subset \operatorname{supp} \bar{\nu}$. With this definition, the analysis of the previous paragraph makes it obvious that dispersion implies weak dispersion.

To verify that the set of dispersed distributions is dense, consider any $\bar{\nu} \in \mathcal{M}$ and any $\bar{\theta} \in \operatorname{supp} \bar{\nu}$. Define the nested sequence of compact sets $V_n := \Theta_{1/n}(\bar{\theta})$ $(n = 1, 2, \ldots)$, each of which can be regarded as a subset of $\Re^m_+$. Let μ denote Lebesgue measure on $\Re^m$. Then one can define $\nu^*_n \in \mathcal{M}$ so that $\nu^*_n(K) := \mu(K \cap V_n)/\mu(V_1)$ for each measurable $K \subset \Theta_S$. Next, construct the distributions $\bar{\nu}_n := [1 - \nu^*_n(V_n)]\, \bar{\nu} + \nu^*_n \in \mathcal{M}$ for $n = 1, 2, \ldots$. Then $\operatorname{supp} \bar{\nu}_n = \operatorname{supp} \bar{\nu} \cup V_n \supset \Theta_{1/n}(\bar{\theta})$, so $\bar{\nu}_n$ must be dispersed. As $n \to \infty$, the Hausdorff distance in Θ_S between $\operatorname{supp} \bar{\nu}_n$ and $\operatorname{supp} \bar{\nu}$ evidently converges to 0. Also, $\mu(V_n) \to 0$ as $n \to \infty$. Therefore, given any bounded continuous function $\phi : \Theta_S \to \Re$, one must have

$$\int_{\Theta_S} \phi(\theta) d\bar{\nu}_n = [1 - \nu^*_n(V_n)] \int_{\Theta_S} \phi(\theta) d\bar{\nu} + \frac{1}{\mu(V_1)} \int_{V_n} \phi(\theta) d\mu \to \int_{\Theta_S} \phi(\theta) d\bar{\nu}$$

Hence, $\bar{\nu}_n$ must converge to $\bar{\nu}$ in the topology of weak convergence, and so in the topology we are using. Thus, the set of distributions $\bar{\nu}$ satisfying A.2 is dense as well as open in $\mathcal{M}$, so is generic.

4. Constructing a Sequence of Mechanisms

4.1. An Expanding Sequence of Random Economies

For $n = 2, 3, \ldots$, consider any economy with the finite set $I_n := \{ 0, 1, 2, \ldots, n \}$ of $n + 1$ agents. In order to describe a general non-anonymous mechanism in this economy, it will be necessary to consider the entire profile θ^{I_n} of different agents' types, rather than just the distribution ν_n. In fact, it will be assumed that θ^{I_n} is a collection of $n + 1$ independent and identically distributed (i.i.d.) random draws from Θ_S with probability distribution $\bar{\nu}$. Moreover, the profile θ^{I_n} is assumed to be $\theta^{I_{n-1}}$ extended by the additional characteristic θ_n of the $(n + 1)$th person — in other words, we have an *expanding* sequence of economies.

Given θ^{I_n}, let $\nu_n \in \mathcal{M}$ denote the associated empirical distribution defined for every Borel set $K \subset \Theta_S$ by

$$\nu_n(K) := \frac{1}{n+1} \#\{\, i \in I_n \mid \theta_i \in K \,\} = \frac{1}{n+1} \sum\nolimits_{i=0}^{n} 1_K(\theta_i)$$

where each $1_K(\theta_i)$ is the indicator variable satisfying $1_K(\theta_i) = 1$ iff $\theta_i \in K$, and $1_K(\theta_i) = 0$ otherwise. As the mean of $n+1$ i.i.d. bounded random variables, the law of large numbers implies that $\nu_n(K)$ converges almost surely to its expected value $\bar{\nu}(K)$ as $n \to \infty$. In particular, almost surely ν_n will converge weakly to $\bar{\nu}$.[2]

Because the sequence of economies is expanding, one must have $\operatorname{supp} \nu_m \subset \operatorname{supp} \nu_n$ whenever $n > m$. Moreover, since ν_n is constructed through sampling from $\bar{\nu}$, it must be true that $\operatorname{supp} \nu_n \subset \operatorname{supp} \bar{\nu}$. Let V be any open set that intersects $\operatorname{supp} \bar{\nu}$. Then $\bar{\nu}(V) > 0$, implying that $\nu_n(V) > 0$ for all large n. So V intersects $\operatorname{supp} \nu_n$. It follows that the set $\operatorname{supp} \bar{\nu} \setminus \cup_{n=1}^{\infty} \operatorname{supp} \nu_n$ has empty interior, so $\cup_{n=1}^{\infty} \operatorname{supp} \nu_n$ must have $\operatorname{supp} \bar{\nu}$ as its closure. The Hausdorff distance between $\operatorname{supp} \nu_n$ and $\operatorname{supp} \bar{\nu}$ must therefore converge to 0 as $n \to \infty$. It follows that $\nu_n \to \bar{\nu}$ in the topology of $\mathcal{M}$ we are using.

4.2. Budget Constraints and the Balancing Agent

For each $n = 2, 3, \ldots$ and each random type profile θ^{I_n}, we will construct an allocation mechanism $F_n(\theta^{I_n}) = \langle F_{n,i}(\theta^{I_n}) \rangle_{i \in I_n}$. As mentioned above, these mechanisms will be "non-parametric", in the sense that they depend only on agents' revelations concerning the unknown profile θ^{I_n}; no information about the limiting distribution $\bar{\nu}$ is used in their construction.

Each mechanism F_n of the sequence will single out agent 0 to play a special balancing role. Also, the remaining n agents will be divided into the two approximately equally sized groups $I_n^1 = \{\, 1, 2, \ldots, r_n \,\}$ and $I_n^2 = \{\, r_n{+}1, \ldots, n \,\}$, where r_n ($n = 2, 3, \ldots$) is any sequence of integers such that $n^{-1} r_n \to \frac{1}{2}$ as $n \to \infty$. Sometimes we will write $r_n^j = \# I_n^j$ ($j = 1, 2$), implying that $r_n^1 = r_n$ and $r_n^2 = n - r_n$. Given the type profile θ^{I_n}, let $\theta^{I_n^1}$ and $\theta^{I_n^2}$ denote the type profiles of the two groups I_n^1 and I_n^2, with corresponding empirical distributions $\nu_n^1, \nu_n^2 \in \mathcal{M}$ respectively.

[2] Because Θ_S is a separable metric space, this accords with the Glivenko–Cantelli theorem in the form due to Parthasarathy [29, Theorem 7.1, p. 53], also cited by Hildenbrand [18, pp. 52–53]. But the result we need is slightly stronger because the supports must also converge.

The measurable selection $p^W(\cdot)$ of A.1 will be used to confront agents $i \in I_n^1$ with the same budget constraint which depends on ν_n^2, the distribution of characteristics for individuals in the complementary set I_n^2. Similarly, the mechanism confronts the agents $i \in I_n^2$ with a different budget constraint which depends on ν_n^1, the distribution of characteristics for individuals in I_n^1. Both budget constraints involve separate transactions fees for buying and selling all except the *numéraire* commodity 1. These fees are assumed to be quadratic in order to preserve smoothness of individual demand functions. In the later formal analysis, this will permit the use of linear approximations to agents' demand functions.

Specifically, all except the balancing agent 0 will face one of two price vectors $p \in \Delta^0$ and one of two transaction fees $\tau > 0$. Then each p and τ jointly determine a non-linear budget set

$$B(p, \tau) := \{ x \in \Re^G \mid p\, x + \tau \, \|x_{-1}\|^2 \leq 0 \}$$

Of course, because $\tau > 0$, the budget set $B(p, \tau)$ is convex.

Next, for each possible agent type $\theta \in \Theta_S$, define the utility-maximizing net trade vector

$$x(p, \tau; \theta) := \arg\max_x \{ u(x; \theta) \mid x \in X(\theta) \cap B(p, \tau) \}$$

Obviously, when $\tau = 0$ the budget set $B(p, 0)$ becomes the Walrasian budget set $B^W(p, 0)$, implying that $x(p, 0; \theta) = x^W(p, 0; \theta)$ for each $\theta \in \Theta_S$.

4.3. Budget Constraints

Smoothness Proposition. *Under A.2, there exists $\epsilon > 0$ such that the utility-maximizing demand function $(p, \tau, \theta) \mapsto x(p, \tau; \theta)$ is jointly continuous in (p, τ, θ) and continuously differentiable in (p, τ) on the domain D of (p, τ, θ) satisfying $\|p - \bar{p}\| < \epsilon$, $0 \leq \tau < \epsilon$ and $\theta \in \operatorname{supp} \bar{\nu}$, with partial Jacobian matrix and partial gradient vector denoted by*

$$x'_p(p, \tau; \theta) = \left(\frac{\partial x_g}{\partial p_h}(p, \tau; \theta) \right)_{g,h \in G \setminus \{1\}} \quad \text{and} \quad x'_\tau(p, \tau; \theta) = \left\langle \frac{\partial x_g}{\partial \tau}(p, \tau; \theta) \right\rangle_{g \in G \setminus \{1\}}$$

where $x'_p(p, 0; \theta)$ is continuous in (p, θ) and uniformly bounded on D.

Proof: By the definition of smooth characteristics in Section 2.1, for all $\theta \in \Theta_S$, all $p \in \Delta^0$, and all small enough $\tau \geq 0$, the net trade vector $x(p, \tau; \theta)$ is "a regular point of demand" in the interior of $X(\theta)$. So the conclusion follows from applying the implicit function theorem to the total derivative of the first-order conditions, as in [24, Prop. 2.7.2]. ∎

Now for each n, given the two empirical distributions ν_n^k and the associated Walrasian equilibrium normalized price vectors $p_n^k := p^W(\nu_n^k)$ $(k = 1, 2)$, define

$$S_n^k = S_n^k(\nu_n^k) := \frac{1}{r_n^k} \sum\nolimits_{i \in I_n^k} \sum\nolimits_{g \in G \setminus \{1\}} \left[x_g(p_n^k, 0; \theta_i) \right]^2$$
$$X_p'(\nu_n^k) := \frac{1}{r_n^k} \sum\nolimits_{i \in I_n^k} x_p'(p_n^k, 0; \theta_i) \quad \text{and} \quad x_\tau'(\nu_n^k) := \frac{1}{r_n^k} \sum\nolimits_{i \in I_n^k} x_\tau'(p_n^k, 0; \theta_i)$$

Next, let e_{-1} denote the $(\ell - 1)$-dimensional vector $(1, 1, \ldots, 1) \in \Re^{G \setminus \{1\}}$. Then, given the two empirical distributions ν_n^k $(k = 1, 2)$, define the associated *transaction fees* $\tau_n^k > 0$ and *price adjustments* $v_n^k \in \Re^{g \in G \setminus \{1\}}$ by

$$\tau_n^k = \tau(\nu_n^k) := \begin{cases} \sum_{g \in G} p_g(\nu_n^k) / S_n^k(\nu_n^k) & \text{if } S_n^k(\nu_n^k) \neq 0 \\ 1 & \text{if } S_n^k(\nu_n^k) = 0 \end{cases}$$

$$v_n^k = v(\nu_n^k) := \begin{cases} \left(0, -[X_p'(\nu_n^k)]^{-1} \left[e_{-1} + \tau_n^k \, x_\tau'(\nu_n^k)\right]\right) & \text{if } X_p'(\nu_n^k) \text{ is invertible} \\ 0 & \text{otherwise} \end{cases}$$

The price adjustment vector v_n^k will be zero if p_n^k is a critical equilibrium, which is only possible if ν_n^k is a singular distribution. This will almost surely not occur for n large enough, because $\bar{\nu} \in \mathcal{M}' \subset \mathcal{M}^*$ and so all distributions near the limiting regular distribution $\bar{\nu}$ must be regular. Note that the mechanism is well defined over the whole domain $\mathcal{M}$.

The specification of the price adjustments and transaction fees is intended to ensure that, for n large enough, with high probability both groups of agents I_n^1 and I_n^2 will generate an excess supply of every commodity. Concretely, choose any sequence $\{\lambda_n\}_{n=1,2,\ldots}$ of positive scalars such that $\lambda_n \to 0$ and yet ${\lambda_n}^2 \, n / \ln n \to \infty$ as $n \to \infty$ — for example, $\lambda_n = \lambda \, n^{\rho - 1/2}$ for some constants λ, ρ satisfying $\lambda > 0$ and $0 < \rho < \frac{1}{2}$. Given each sub-distribution ν_n^k $(k = 1, 2)$, the vectors of Walrasian equilibrium prices p_n^k, price adjustments v_n^k, and transaction fees τ_n^k, every agent in the complementary group I_n^j $(j \neq k)$ will be confronted with the budget set

$$B_n^j(\nu_n^k) = B(p_n^k + \lambda_n \, v_n^k, \lambda_n \, \tau_n^k) = \{ \, x \in \Re^G \mid (p_n^k + \lambda_n \, v_n^k) \, x + \lambda_n \, \tau_n^k \, \|x_{-1}'\|^2 \leq 0 \, \}$$

Note that $B_n^j(\nu_n^k)$ depends only on n and on the distribution ν_n^k of types in the other group I_n^k $(k \neq j)$; it is entirely independent of $\theta^{I_n^j}$. In fact, $B_n^j(\nu_n^k)$ is the Walrasian equilibrium net trade budget set that would arise in the economy with distribution ν_n^k, except that it is corrected by the small adjustments $\lambda_n \, v_n^k$ in the price vector and by the quadratic transaction charges that are also proportional to λ_n. The purpose of the price adjustments is to create negative excess demand, after correcting for the effects of the transaction fees. The precise choices of v_n^k and τ_n^k will be justified in Section 5, in the course of proving Lemma 5.

For each empirical distribution ν_n^k $(k = 1, 2)$, define for each possible agent type $\theta \in \Theta$ the appropriate utility maximizing net trade

$$x_n(\nu_n^k; \theta) := x(p_n^k + \lambda_n \, v_n^k, \lambda_n \, \tau_n^k; \theta)$$

and

$$x_n^0(\nu_n^1, \nu_n^2) := -\sum\nolimits_{i \in I_n^1} x_n(\nu_n^2; \theta_i) - \sum\nolimits_{i \in I_n^2} x_n(\nu_n^1; \theta_i)$$

as the net trade vector required to balance the resulting net trades of the n agents. Then

$$\hat{x}_n^0(\nu_n^1, \nu_n^2; \theta_0) := \begin{cases} 0 & \text{if } x_n^0(\nu_n^1, \nu_n^2) \notin X(\theta_0) \text{ or } u(x_n^0(\nu_n^1, \nu_n^2); \theta_0) < u(0; \theta_0) \\ x_n^0(\nu_n^1, \nu_n^2) & \text{otherwise,} \end{cases}$$

represents the balancing agent 0's preferred net trade vector, given the choice between $x_n^0(\nu_n^1, \nu_n^2)$ and autarky.

4.4. *A Sequence of Mechanisms*

For all $i \in I_n^j$ where $j \neq k$ $(k = 1, 2)$ and for $i = 0$, the *sequence of mechanisms* F_n for $n = 1, 2, \ldots$ is defined as follows:

$$F_{n,i}(\nu_n^1, \nu_n^2; \theta_i, \theta_0) := \begin{cases} x_n(\nu_n^k; \theta_i) & \text{if } \hat{x}_n^0(\nu_n^1, \nu_n^2; \theta_0) \neq 0 \text{ or } x_n^0(\nu_n^1, \nu_n^2) = 0, \\ 0 & \text{if } \hat{x}_n^0(\nu_n^1, \nu_n^2; \theta_0) = 0 \neq x_n^0(\nu_n^1, \nu_n^2), \end{cases}$$

$$\text{and} \quad F_{n,0}(\nu_n^1, \nu_n^2; \theta_0) := \hat{x}_n^0(\nu_n^1, \nu_n^2; \theta_0)$$

Note that each mechanism F_n allows agent 0 to choose autarky for everybody if balancing would violate individual rationality. Thus, F_n is individually rational. It is also resource-balanced by construction.

Because $\lambda_n \to 0$ as $n \to \infty$ and each $x(p, \tau; \theta)$ $(\theta \in \Theta_S)$ is a continuous function of (p, τ) at $(p, \tau) = (\bar{p}, 0)$, the net trade vector $x(p_n^k + \lambda_n \, v_n^k, \lambda_n \, \tau_n^k; \theta)$ of each non-balancing agent $i \in I_n^1 \cup I_n^2$ will converge to the Walrasian net trade vector $x(p(\bar{\nu}), 0; \theta) = x^W(\bar{p}; \theta)$ provided that v_n^k and τ_n^k both remain bounded as $n \to \infty$. Thus, the sequence of mechanisms will be *asymptotically Walrasian* in the sense that, except for the balancing agent 0, each agent i's allocation $F_{n,i}(\theta^{I_n})$ converges to the Walrasian allocation $x^W(\bar{p}; \theta_i)$ almost surely because the empirical distributions ν_n^1, ν_n^2 corresponding respectively to $\theta^{I_n^1}, \theta^{I_n^2}$ both converge almost surely to $\bar{\nu}$.

Evidently, truthfulness is always a dominant strategy for the balancing agent 0, who faces either the choice between 0 and $x_n^0(\nu_n^1, \nu_n^2)$, or else no choice at all; in particular, agent 0's option set is independent of θ_0. As for the agents in either group I_n^j ($j = 1, 2$), their common budget set $B_n^j(\nu_n^k)$ ($k \neq j$) is independent of all their types. However, in some cases, the mechanism F_n grants them the power as individuals to determine whether the economy is reduced to autarky or not. In fact, given any particular characteristic profile θ^{I_n}, truthfulness is always a dominant strategy for all $i \in I_n^1 \cup I_n^2$ unless the mechanism generates autarky. This situation could arise whenever the balancing agent strictly prefers autarky to the balancing net trade vector. Even then, an individual agent $h \in I_n^j$ of type θ_h can only manipulate advantageously if an alternative type $\eta_h \in \Theta_S$ can be found such that $u\left(x_n^h(\nu_n^k; \eta_h); \theta_h\right) > u(0; \theta_h)$ and also

$$u\left(x_n^0(\tilde{\nu}_n^j(\eta_h), \nu_n^k; \theta_0\right) \geq u(0; \theta_0) > u\left(x_n^0(\nu_n^1, \nu_n^2); \theta_0\right)$$

where $\tilde{\nu}_n^j(\eta_h)$ denotes the overall distribution of characteristics reported by agents in I_n^j when θ_h is replaced by η_h.

Thus, according to the definition given in Section 1.4, the mechanism F_n may not be locally strategy-proof for some profiles of individual characteristics. But this breakdown cannot occur unless the balancing agent strictly prefers autarky to the balancing trade prescribed by the mechanism. Under the above assumptions, it will be proved that, as $n \to \infty$, there is a zero limiting probability that the allocation determined by F_n is autarkic. Intuitively, by introducing quadratic transaction fees and then adjusting the equilibrium prices slightly, the equilibrium gets perturbed in a way that generates an expected surplus of every commodity. Moreover, this surplus becomes large as the economy becomes large, even though the expected surplus per head vanishes in the limit. In this way, even though the prices facing each group may be only approximately market clearing for that group, it is still possible to ensure that the balancing net trade vector $x_n^0(\nu_n^1, \nu_n^2)$ for agent 0 is non-negative — or even strictly positive — with a probability which tends to 1 as $n \to \infty$. The implication will be that, with probability which converges to 1, the balancing net trade vector will be individually rational for agent 0. The sequence of mechanisms will then be asymptotically strategy-proof, as well as resource-balanced, individually rational, and asymptotically Walrasian.

Note that the condition defined above for truthfulness not to be an optimal strategy for a given agent — whenever all other agents are assumed to report truthfully — is met with considerably lower probability than the condition required for the existence of a surplus in every commodity. For this reason, the estimates we present below for autarky to be avoided overstates the probability of advantageous manipulation by a single agent. Note also that, whenever all other agents are assumed to report truthfully, a given agent's incentives to manipulate will decrease rapidly as the size of the economy grows and will eventually disappear. Not only does the benefit from any fixed attempt to manipulate tend to zero, as shown by Roberts and Postlewaite [31], but also its expected value becomes negative for large enough economies (under the expected utility hypothesis). For a given agent, the cost of an unnecessary attempt to manipulate is an inappropriate allocation — namely, a net trade vector that would be optimal for the distorted characteristic instead of the true characteristic; this happens whenever the economy stays away from autarky, which occurs with a probability that converges to 1 whenever all other agents report truthfully; moreover, for any fixed attempt to manipulate, this cost is independent of n. On the other hand, each agent's benefit from successful manipulation — i.e., the possibility of steering the economy away from autarky — will materialize with a probability that converges to zero as the economy grows infinitely large, provided that all other agents report truthfully. So, in this case, the expected value of any fixed attempt to manipulate will become negative for large enough economies. This ensures that, if a given agent expects everyone else to report truthfully, the best response will converge to truthful revelation. Obviously, this does not imply that every sequence of Bayesian equilibria converges to truthful revelation.

5. Formal Proofs

The previous section presented only a very informal argument. This section contains a more rigorous treatment based upon Bhattacharya and Majumdar's [6] approach to a somewhat similar problem — see also Weller [36].

Main Theorem. *Under A.1–A.2, each mechanism* $F_n(\theta^{I_n})$ *is well defined as a function of the list* θ^{I_n} *of* $n+1$ *i.i.d. random characteristics drawn from the distribution* $\bar{\nu} \in \mathcal{M}'$*. Also, except for the balancing agent 0, each agent i's allocation* $F_{n,i}(\theta^{I_n})$ *generated by each mechanism converges to the Walrasian allocation* $x_i^W(\bar{p};\theta_i)$ *almost surely as* $n \to \infty$*. Furthermore, there exists a constant* $\delta > 0$ *depending only on the distribution* $\bar{\nu}$ *such that, for all large n, the mechanism* $F_n(\theta^{I_n})$ *is always balanced and individually rational, and is also locally strategy-proof at* θ^{I_n} *with probability at least* $1 - \delta\, n^{-1/2}\,(\ln n)^{-3/2}$*.*

The proof will proceed through a series of lemmas. First, note that $p_n^k \to \bar{p}$ as $n \to \infty$ because Section 4.1 shows that $\nu_n^k \to \bar{\nu}$ almost surely, whereas $\bar{\nu} \in \mathcal{M}'$ and $\lambda_n \to 0$ by definition. It follows from the Smoothness Proposition of Section 4.3 that $F_{n,i}(\theta^{I_n}) \to x(p(\bar{\nu}), 0; \theta_i) = x^W(\bar{p};\theta_i)$ almost surely for all $i \in I_n^1 \cup I_n^2$. In this sense, the mechanism will be asymptotically Walrasian. The sixth and last lemma shows that the balancing agent 0's net trade vector satisfies $x_n^0(\nu_n^1, \nu_n^2) \geqq 0$ with probability at least $1 - \delta\, n^{-1/2}\,(\ln n)^{-3/2}$ for a suitable positive constant $\delta > 0$. From this, the above theorem follows immediately. Finally, because the set of distributions satisfying A.2 is open and dense in the domain $\mathcal{M}$, the properties of individual rationality, asymptotic efficiency and asymptotic strategy-proofness hold generically.

The first lemma is a useful result in mathematical statistics due to Bhattacharya and Rao [7, Corollary 17.13], also cited by Weller [36, p. 75], as applied to a sequence of random variables that are not only independent, but also identically distributed.

Lemma 1. *Let* X_n $(n = 1, 2, \ldots)$ *be a sequence of* $\Re^G$*-valued i.i.d. random variables with common zero mean, variance–covariance matrix V, and sth absolute moments* $\mathbb{E}\,||X||^s$ $(s = 1, 2, \ldots)$*. Let* γ *and* Γ *respectively denote the smallest and largest eigenvalues of V. Suppose that* $\gamma > 0$ *and that* $\mathbb{E}\,||X||^s < \infty$ *for some integer* $s \geq 3$*. Then there exists a* *uniformly bounded sequence* δ_n $(n = 1, 2, \ldots)$ *such that*

$$\text{Prob}\left(\left\| \frac{1}{n} \sum\nolimits_{j=1}^{n} X_j \right\|^2 > \frac{1}{n}(s-1)\,\Gamma\,\ln n \right) \leq \delta_n\, n^{-(s-2)/2}\,(\ln n)^{-s/2}$$

From now on, A.1–A.2 are postulated throughout.

LEMMA 2. *Let* Λ *be the largest eigenvalue of the variance–covariance matrix of the random Walrasian net trade vector* $x^W_{-1}(\bar{p};\theta)$*. Then there exists* $\delta_0 > 0$ *such that*

$$\text{Prob}\left(\left\| \frac{1}{r} \sum\nolimits_{i=1}^{r} x_{-1}\big(p(\bar{\nu}), 0; \theta_i\big) \right\|^2 > \frac{2}{r^2}\,\Lambda\,\ln r \right) \leq \delta_0\, r^{-1/2}\,(\ln r)^{-3/2}$$

PROOF: Following the argument of Weller [36], this is a direct application of Lemma 1 to the infinite sequence $\{x^W_{-1}(\bar{p};\theta_i)\}_{i=1,2,\ldots}$ of i.i.d. random vectors, with $s = 3$ and $\delta_0 = \sup_r \{\delta_r\}$. Indeed, each member of the sequence $\{x^W(\bar{p};\theta_i)\}$ belongs to the compact set $\{\, x \in \Re^G \mid x \geqq \underline{x} \text{ and } \bar{p}\, x \leq 0 \,\}$. So every absolute moment of the distribution of $x^W_{-1}(\bar{p};\theta)$ is finite. Finally, because of A.2, each eigenvalue of the common variance–covariance matrix of the net trade vectors is positive. ∎

The following two lemmas both make use of the notation $t^k_n := p^k_n - \bar{p}$.

LEMMA 3. *There exist constant positive scalars* d^k_1 *and* δ^k_1 $(k = 1, 2)$ *such that*

$$\text{Prob}\left(\|t^k_n\|^2 > (d^k_1/r^k_n)\,\ln r^k_n \right) \leq \epsilon^k_{1,n}$$

where $\epsilon^k_{1,n} := \delta^k_1\,(r^k_n)^{-1/2}\,(\ln r^k_n)^{-3/2}$.

PROOF: By construction, each sequence $p^k_n = p^W(\nu^k_n)$ is taken from the measurable selection $p^W(\cdot)$ from the Walrasian equilibrium price correspondence defined on the whole domain $\mathcal{M}$. Clearly, this gives a sequence of measurable equilibrium random prices. Because of A.2, each eigenvalue of the common variance–covariance matrix of the price derivatives of the net trade vectors is positive. So the result follows directly from [36, Theorem 2]. ∎

Lemma 4. *There exist constant positive scalars d_2 and δ_2 such that*

$$\mathrm{Prob}\left(\left\| \frac{1}{r_n^j} \sum\nolimits_{i \in I_n^j} x(p_n^k, 0; \theta_i) \right\| > d_2\, n^{-1/2}\, (\ln n)^{1/2} \right) \leq \delta_2\, n^{-1/2}\, (\ln n)^{-3/2}$$

for all large n, and for $j, k = 1, 2$ with $j \neq k$.

Proof: Because $\bar{\nu}$ has compact support, the function $\psi_n^j(\zeta) := \sum_{i \in I_n^j} x_{-1}(\bar{p} + \zeta\, t_n^k, 0; \theta_i)$ of the single real variable ζ satisfies the assumptions of the mean value theorem. So, for each n, there exists $\zeta_n^j \in [0, 1]$ such that $\psi_n^j(1) = \psi_n^j(0) + \psi_n^{j\prime}(\zeta_n^j)$ and so

$$\sum\nolimits_{i \in I_n^j} x_{-1}(p_n^k, 0; \theta_i) = \sum\nolimits_{i \in I_n^j} \left[x_{-1}(\bar{p}, 0; \theta_i) + x'_p(\bar{p} + \zeta_n^j\, t_n^k, 0; \theta_i)\, t_n^k \right]$$

Now define the two vectors

$$a_n^j := \frac{1}{r_n^j} \sum\nolimits_{i \in I_n^j} x_{-1}(\bar{p}, 0; \theta_i) \quad \text{and} \quad b_n^j := \frac{1}{r_n^j} \sum\nolimits_{i \in I_n^j} x'_p(\bar{p} + \zeta_n^j\, t_n^k, 0; \theta_i)\, t_n^k$$

in $\Re^{G \setminus \{1\}}$. Then obviously

$$\frac{1}{r_n^j} \sum\nolimits_{i \in I_n^j} x_{-1}\left(p_n^k, 0; \theta_i\right) = a_n^j + b_n^j$$

By Lemma 2, for $j = 1, 2$ there exists $d_3^j > 0$ such that $||a_n^j|| > d_3^j (r_n^j)^{-1}\, (\ln r_n^j)^{1/2}$ with probability less than $\epsilon_n := \delta_0\, (r_n^j)^{-1/2}\, (\ln r_n^j)^{-3/2}$.

Because $\bar{\nu}$ has compact support, the continuous partial Jacobian matrix $x'_p(p, 0; \theta)$ is uniformly bounded for all p in some neighbourhood of $\bar{p}$ and for all θ in the compact set supp $\bar{\nu}$. Also, the price sequence p_n^k converges almost surely to $\bar{p}$, so t_n^k converges almost surely to 0. Hence, for large n, almost surely there exists ξ_n such that $\left\| x'_p(\bar{p} + \zeta\, t_n^k, 0; \theta_i) \right\| \leq \xi_n$ for all $\zeta \in [0, 1]$ and all $i \in I_n^j$, so $||b_n^j|| \leq \xi_n\, ||t_n^k||$. Therefore, by Lemma 3, if $d_4^j := \xi_n\, (d_1^j)^{1/2}$, then $||b_n^j|| > d_4^j (r_n^j)^{-1/2}\, (\ln r_n^j)^{1/2}$ with probability less than $1 - \epsilon_{1,n}^k$.

The previous two paragraphs imply that, with probability no less than $1 - \epsilon_n - \epsilon_{1,n}^k$, one has both $||a_n^j|| \leq d_3^j\, (r_n^j)^{-1}\, (\ln r_n^j)^{1/2}$ and $||b_n^j|| \leq d_4^j\, (r_n^j)^{-1/2}\, (\ln r_n^j)^{1/2}$. But $r_n^j / n \to \frac{1}{2}$ as $n \to \infty$. Hence, given any $d_5 > \sqrt{2}\, \max\{\, d_4^1, d_4^2\,\}$, for large n one has $||a_n^j + b_n^j|| > d_5\, n^{-1/2}\, (\ln n)^{1/2}$ with probability no greater than $\epsilon_2 = (\delta_0 + \delta_1)\, n^{-1/2}\, (\ln n)^{-3/2}$. Also, the budget constraints imply that

$$\frac{1}{r_n^j} \sum\nolimits_{i \in I_n^j} x_1\, (\bar{p}, 0; \theta_i) = -p_{-1}(\bar{\nu})\, (a_n^j + b_n^j).$$

The result follows with $d_2 := [1 + ||p_{-1}(\bar{\nu})||\,]\, d_5$ and with $\delta_2 := \delta_0 + \delta_1$. ∎

Recall the notation $e = (1, 1, \ldots, 1) \in \Re^G$, $e_{-1} = (1, 1, \ldots, 1) \in \Re^{G\setminus\{1\}}$, and

$$S_n^k(\nu_n^k) = \frac{1}{r_n^k} \sum\nolimits_{i \in I_n^k} \sum\nolimits_{g \in G\setminus\{1\}} \left[x_g(p_n^k, 0; \theta_i)\right]^2.$$

Denote also

$$x_n^k(\lambda; v, \tau) := \frac{1}{r_n^k} \sum\nolimits_{i \in I_n^k} x\left(p_n^k + \lambda\, v, \lambda\, \tau; \theta_i\right) \quad \text{and} \quad \bar{S} := \int_{\Theta_S} \sum\nolimits_{g \in G\setminus\{1\}} \left[x_g\left(\bar{p}, 0; \theta\right)\right]^2 d\bar{\nu}$$

Lemma 5. *For all large n, one has* $x_n^k{}'(0; v, \tau) = \left.\dfrac{d}{d\lambda} x_n^k(\lambda; v, \tau)\right|_{\lambda=0} = -e$ *almost surely when* $v = v_n^k$ *and* $\tau = \tau_n^k$.

Proof: Because the price sequence p_n^k converges almost surely to $\bar{p}$, the smoothness proposition of Section 4.3 implies that

$$X_p'(\nu_n^k) := \frac{1}{r_n^k} \sum\nolimits_{i \in I_n^k} x_p'(p_n^k, 0; \theta_i) \quad \text{and} \quad x_\tau'(\nu_n^k) := \frac{1}{r_n^k} \sum\nolimits_{i \in I_n^k} x_\tau'(p_n^k, 0; \theta_i)$$

are almost surely well defined for all large n. Now write v in the partitioned form $(0, v_{-1})$. Then, after omitting good 1 from the gradient vector $x_n^k{}'(0; v, \tau) = \left.\dfrac{d}{d\lambda} x_n^k(\lambda; v, \tau)\right|_{\lambda=0}$, the remaining $\ell - 1$ components evidently satisfy

$$x_{n,-1}^k{}'(0; v, \tau) = X_p'(\nu_n^k)\, v_{-1} + \tau\, x_\tau'(\nu_n^k)$$

Now, by smoothness and A.1–A.2, the matrix $X_p'(\nu_n^k)$ converges almost surely to $X_p'(\bar{\nu})$. By A.2, $\bar{\nu}$ is regular, so $X_p'(\bar{\nu})$ is invertible. Hence, $X_p'(\nu_n^k)$ is almost surely invertible when n is large enough. Then one can ensure that $x_{n,-1}^k{}'(0; v, \tau) = -e_{-1}$ by choosing $v = (0, v_{-1}(\nu_n^k))$ where

$$v_{-1}(\nu_n^k) := -[X_p'(\nu_n^k)]^{-1}\, [e_{-1} + \tau\, x_\tau'(\nu_n^k)]$$

Next, note that $x_n^k(0; v, \tau) = 0$ because p_n^k is a Walrasian equilibrium price vector for the distribution ν_n^k. Also, taking the average of the budget constraints for the agents $i \in I_n^k$ implies that

$$x_{n,1}^k(\lambda; v, \tau) = -[p_{-1}^W(\nu_n^k) + \lambda\, v_{-1}]\, x_{n,-1}^k(\lambda; v, \tau) \\ - \frac{\lambda}{r_n^k} \sum\nolimits_{i \in I_n^k} \tau \sum\nolimits_{g \in G\setminus\{1\}} \left[x_g(p_n^k + \lambda\, v, \lambda\, \tau; \theta_i)\right]^2$$

Then, because $x_{n,-1}^k(0; v, \tau) = 0$, differentiating w.r.t. λ at $\lambda = 0$ gives

$$x_{n,1}^k{}'(0; v, \tau) = -p_{-1}^W(\nu_n^k)\, x_{n,-1}^k{}'(0; v, \tau) - \tau\, S_n^k(\nu_n^k) = p_{-1}^W(\nu_n^k)\, e_{-1} - \tau\, S_n^k(\nu_n^k)$$

Now, A.2 implies that $\bar{S} > 0$. Because $\nu_n^k \to \bar{\nu}$ almost surely and so $p_n^k \to \bar{p}$, whereas $\operatorname{supp} \bar{\nu}$ is compact, it follows that $x(p_n^k, 0; \theta)$ tends to $x(\bar{p}, 0; \theta)$ uniformly for all $\theta \in \operatorname{supp} \bar{\nu}$. Therefore $S_n^k \to \bar{S}$ almost surely. This implies in particular that $S_n^k > 0$ almost surely for large n. Hence, choosing $\tau = [1 + p_{-1}^W(\nu_n^k)\, e_{-1}]/S_n^k(\nu_n^k)$ ensures that $x_{n,1}^k{}'(0; v, \tau) = -1$. Recalling that $p_1^W(\nu_n^k) = 1$, it follows that choosing

$$\tau_n^k := \sum\nolimits_{g \in G} p_g(\nu_n^k)/S_n^k(\nu_n^k) \quad \text{and} \quad v_n^k := \left(0, -[X_p'(\nu_n^k)]^{-1}\, [e_{-1} + \tau_n^k\, x_\tau'(\nu_n^k)]\right)$$

does ensure that $x_n^k{}'(0; v_n^k, \tau_n^k) = -e$. ∎

LEMMA 6. *For large n one has* $\operatorname{Prob}\left(x_n^0(\nu_n^1, \nu_n^2) \geqq 0\right) \geq 1 - 2\,\delta_2\, n^{-1/2}\, (\ln n)^{-3/2}$.

PROOF: By definition, $x_n^0(\nu_n^1, \nu_n^2) = -\sum_{j=1}^2 \sum_{i \in I_n^j} x_n(\nu_n^k; \theta_i)$ where $x_n(\nu_n^k; \theta_i)$ denotes $x(p_n^k + \lambda_n\, v_n^k, \lambda_n\, \tau_n^k; \theta_i)$. Now, expanding in a first-order Taylor series about $\lambda = 0$ gives

$$\frac{1}{r_n^j} \sum\nolimits_{i \in I_n^j} x_n(\nu_n^k; \theta_i) = c_n^j + \lambda_n\, y_n^j + o(\lambda_n)$$

where

$$c_n^j := \frac{1}{r_n^j} \sum\nolimits_{i \in I_n^j} x(p_n^k, 0; \theta_i); \quad y_n^j := \frac{1}{r_n^j} \sum\nolimits_{i \in I_n^j} \frac{d}{d\lambda} x(p_n^k + \lambda\, v_n^k, \lambda\, \tau_n^k; \theta_i)\bigg|_{\lambda = 0}$$

Excluding the *numéraire* commodity 1, Lemma 5 implies that

$$y_{n,-1}^j = X_p'(\nu_n^j)\, v_{-1}(\nu_n^k) + \tau_n^k\, x_\tau'(\nu_n^j)\, e_{-1}$$

Because the construction in Section 4.3 implies that $X_p'(\nu_n^k)\, v_{-1}(\nu_n^k) = -[e_{-1} + \tau_n^k\, x_\tau'(\nu_n^k)]$, it follows that

$$y_{n,-1}^j = -e_{-1} + [X_p'(\nu_n^j) - X_p'(\nu_n^k)]\, v_{-1}(\nu_n^k) + \tau_n^k\, [x_\tau'(\nu_n^j) - x_\tau'(\nu_n^k)]$$

Almost surely as $n \to \infty$, the matrices $X_p'(\nu_n^j)$ and $X_p'(\nu_n^k)$ both converge to $X_p'(\bar{\nu})$, whereas $x_\tau'(\nu_n^j)$ and $x_\tau'(\nu_n^k)$ both converge to $x_\tau'(\bar{\nu})$; also $\tau_n^k \to \bar{\tau} := \sum_{g \in G} p_g(\bar{\nu})/\bar{S}$, and finally $v_{-1}(\nu_n^k) \to \bar{v}_{-1} := -[X_p'(\bar{\nu})]^{-1}\, [e_{-1} + \bar{\tau}\, x_\tau'(\bar{\nu})]$. It follows that $y_{n,-1}^j \to -e_{-1}$ as $n \to \infty$. Then budget exhaustion implies that, almost surely,

$$\begin{aligned} y_{n,1}^j &= -p_{-1}^W(\nu_n^k)\, y_{n,-1}^j - \frac{\tau_n^k}{r_n^j} \sum\nolimits_{i \in I_n^j} \sum\nolimits_{g \in G \setminus \{1\}} \left[x_g(p_n^k, 0; \theta_i)\right]^2 \\ &\to p_{-1}^W(\nu_n^k)\, e_{-1} - \bar{\tau}\, \bar{S} = -1 \end{aligned}$$

This shows that $y_n^j \to -e$ almost surely. Hence, $y_n^j \leqq -\frac{1}{2} e$ almost surely for large n, and so

$$\frac{1}{r_n^j} \sum\nolimits_{i \in I_n^j} x_n(\nu_n^k; \theta_i) \leqq c_n^j - \tfrac{1}{2}\, \lambda_n\, e + o(\lambda_n)$$

implying that

$$\frac{1}{n} x_n^0(\nu_n^1, \nu_n^2) \geqq -\frac{r_n^1}{n} c_n^1 - \frac{r_n^2}{n} c_n^2 + \tfrac{1}{2}\, \lambda_n\, e + o(\lambda_n)$$

But now, by Lemma 4, there is a probability of at least $1 - 2\delta_2\, n^{-1/2}\, (\ln n)^{-3/2}$ that for $j = 1, 2$ one has $||c_n^j|| \leq d_2\, n^{-1/2}\, (\ln n)^{1/2}$ and so $c_n^j \leqq -d_2\, n^{-1/2}\, (\ln n)^{1/2}\, e$. Also, r_n^1/n and r_n^2/n both tend to $\frac{1}{2}$ as $n \to \infty$. Because $\lambda_n > 2d_2\, n^{-1/2}\, (\ln n)^{1/2}$ for n large enough, the result follows. ▮

6. Conclusion

6.1. Three Alternative Mechanisms Compared

Instead of each mechanism F_n defined in Section 4, consider the two alternative mechanisms F_n^1 and F_n^2 defined by:

$$F_{n,i}^1(\nu_n^1, \nu_n^2; \theta_i) := x_n(\nu_n^k; \theta_i) \quad \text{and} \quad F_{n,0}^1(\nu_n^1, \nu_n^2; \theta_0) := \hat{x}_n^0(\nu_n^1, \nu_n^2; \theta_0);$$

$$F_{n,i}^2(\nu_n^1, \nu_n^2; \theta_i) := x_n(\nu_n^k; \theta_i) \quad \text{and} \quad F_{n,0}^2(\nu_n^1, \nu_n^2) := x_n^0(\nu_n^1, \nu_n^2).$$

For $n = 2, 3, \ldots$, the three mechanisms F_n, F_n^1 and F_n^2 differ only in the role played by the balancing agent 0. First, as stated in Section 4, mechanism F_n allows agent 0 to choose autarky for everybody if balancing would violate individual rationality. Second, mechanism F_n^1 also allows agent 0 to choose 0, but then resource balance may be violated. Finally, mechanism F_n^2 requires agent 0 to balance other agents' net trades whether or not this is individually rational or even feasible.

Clearly, all three mechanisms will be asymptotically Walrasian. Moreover, the proof of the main theorem presented in Section 5 implies that:

(a) $F_n^1(\theta^{I_n})$ is always individually rational and strategy-proof; it is also resource-balanced with probability at least $1 - \delta\, n^{-1/2}\, (\ln n)^{-3/2}$;

(b) $F_n^2(\theta^{I_n})$ is always resource-balanced and strategy-proof; it is also individually rational with probability at least $1 - \delta\, n^{-1/2}\, (\ln n)^{-3/2}$.

Though closely related, these three mechanisms have different advantages and drawbacks. The first, F_n, has the main advantage of always being resource-balanced: the resulting allocation is thus unambiguous. Indeed, because the mechanism is also individually rational, even for the balancing agent, its physical feasibility is guaranteed. Its obvious drawback is that asymptotic strategy-proofness is not quite strategy-proofness.

The mechanism F_n^1 has the advantage of being always strategy-proof. However, because it may be unbalanced with probability close to zero, its true incentive properties may be less favourable. If some rational agents anticipate even a very small probability of imbalance, their behaviour is likely to be influenced by expectations about the future rationing scheme that will ultimately be required to restore balance. Obviously, this could spoil the incentive for truthful reporting.

Finally, the mechanism F_n^2 is resource-balanced and strategy-proof, but may not satisfy the individual rationality constraint $u\left(x_n^0(\nu_n^1, \nu_n^2); \theta_0\right) \geq u(0; \theta_0)$. Indeed, F_n^2 may not be feasible because, for some profiles of individual characteristics, it may be false that $x_n^0(\nu_n^1, \nu_n^2) \in X(\theta_0)$. This is undoubtedly a rather unsatisfactory feature.

In fact, even when the mechanism F_n^1 or F_n^2 breaks down because the balancing agent prefers autarky, mechanism F_n is still quite likely to succeed. This is because F_n is locally strategy-proof except when there is some agent whose demand is so large that moderating only one agent's net trade would allow that agent and the balancing agent to reach a new allocation making both of them better off than under autarky.

Summing up, asymptotically all three mechanisms have strong efficiency, incentive and feasibility properties; however, there is always a small probability of failure that vanishes in the limit as the number of agents becomes infinite. But this probability seems smallest for the mechanism F_n defined in Section 4.

6.2. Better Probability Estimates

Section 5 placed rather crude upper bounds on the probability that any of the three mechanisms would fail. Yet from standard central limit theorems, it is rather intuitive that the allocation received by the balancing agent — which defines the fundamental properties of the mechanisms — should be asymptotically normal. In fact, if we define $\xi_0^n := \xi_0^n(\nu_n^1, \nu_n^2) = \frac{1}{n} x_n^0(\nu_n^1, \nu_n^2)$, then $\xi_0^n = \lambda_n e + c_n + 0(\lambda_n^2)$ where $c_n := \frac{1}{2}(c_n^1 + c_n^2)$. Now we know from Bhattacharya and Majumdar [6] that $\sqrt{n} c_n^1$ and $\sqrt{n} c_n^2$ are asymptotically normal. So therefore is $\sqrt{n} c_n$, since its limit is a linear combination of two normal random variables. It is rather complicated to find the mean and the covariance matrix, however, because c_n^1 and c_n^2 are correlated random variables. So it is not very easy to provide better probability estimates.

6.3. *Optimal Choice of* λ_n

Obviously, if λ_n converges slowly to zero (i.e., if ρ is close to 1/2), then the probability of imbalance shrinks rapidly to zero, but convergence to the limiting Walrasian allocation is slow. The reverse will be true if ρ is close to zero. In the choice of λ_n this faces us with a clear trade-off between incentive and efficiency properties.

6.4. *A "Folk" Mechanism*

Consider the following mechanism. As usual, all $n + 1$ agents are asked to report their characteristics. Then everybody except the balancing agent is assigned a price vector equal to a Walrasian equilibrium price for the economy of $n - 1$ agents without that agent or the balancing agent, and is allocated the corresponding net trade vector. As mentioned by Jackson and Manelli [21, fn. 21, p. 374]: "We have heard this mechanism suggested by many people, but have not been able to find any reference for it. Thus we think of it as a 'folk' mechanism, but are quite happy to stand corrected." (In fact, it was mentioned in [17] at least implicitly.) Whenever the individual prices are close to the same Walrasian price, the resulting allocations would be asymptotically Walrasian. Truthful revelation is clearly a dominant strategy; however, the mechanism generally lacks balance.

Kovalenkov [23] constructs an example in which any specification of the folk mechanism can create an arbitrarily large aggregate imbalance, relative to a uniform bound on endowments, even in large economies. He also proves that the per capita imbalance of the mechanism vanishes generically in large economies. Consequently, for large enough economies in which all possible agents have endowments that are uniformly bounded away from zero by some $\hat{e} \in \Re^G_{++}$, there is an "adjusted" folk mechanism which taxes all n agents the fixed proportion $\epsilon\,\hat{e}$ of the bounding vector $\hat{e}$, and uses these amounts to restore balance after discarding any unused surpluses. This mechanism is also "individually ϵ-rational" (in the sense that no agent with initial endowment e is made worse off than with the reduced endowment $e - \epsilon\,\hat{e}$) and asymptotically "ϵ-Walrasian" (in the sense that the resulting allocation is asymptotically close to a Walrasian allocation for an economy with the reduced endowments $e - \epsilon\,\hat{e}$). Like the mechanism F_n^2 defined in Section 6.1, this adjusted folk mechanism attempts to reconcile resource-balance, strategy-proofness and asymptotic efficiency at the cost of violating individual rationality. However, the mechanism is asymptotically only approximately Walrasian, rather than asymptotically Walrasian. Moreover, this result does not hold generically due to the restriction imposed on endowments.

Clearly, one could reformulate each of our three mechanisms along the lines of the folk mechanism. That is, there could be a balancing agent, as defined above, combined with a different price and transaction fees for each of the other n agents. Apart from the balancing agent, the resulting mechanisms would then be anonymous. Additionally, the probability of failure (of balance, or strategy-proofness, or individual rationality) and the speed of convergence (to a Walrasian allocation) might well be improved. Obviously, the increased costs in terms of computational requirements could be significant as the economy grows.

6.5. Non-Walrasian Mechanisms

An allocation mechanism for an exchange economy is strategy-proof if and only if the allocation to each agent can be decentralized by a budget set that is independent of that agent's characteristics — see for instance Hammond [17, Theorem 1]. In a continuum economy, consider a given non-Walrasian anonymous strategy-proof mechanism that can be decentralized for each $\nu \in \mathcal{M}$ by the budget set $B(\nu) \subset \Re^G$. More specifically, suppose that the mechanism determines $x(\nu, \theta)$ as a function of $\nu \in \mathcal{M}$ and $\theta \in \Theta_S$. Suppose too that

$$x(\nu, \theta) \in \arg\max_x \{ \, u(x; \theta) \mid x \in X(\theta) \quad \text{and} \quad x_1 \leq \beta_1(\nu, x_{-1}) \, \}$$

where, for each $\nu \in \mathcal{M}$, the function $\beta_1(\nu, x_{-1})$ is strictly decreasing in x_{-1}. For example, in the tax models of Vickrey [35] and Mirrlees [27], θ is a skill level, x_{-1} is a scalar measure of labour income, and for each distribution ν of labour skills, the function $\beta_1(\nu, x_{-1})$ specifies allowable consumption expenditure, which is equal to after-tax income.

In the general model with an arbitrary finite set of commodities, suppose that β_1 is also smooth and concave in x_{-1} because $B(\nu)$ is convex and has a smooth frontier. Then arguments similar to those in Section 5 can be applied to the modified budget constraint

$$x_1 + \lambda \, v \, x_{-1} + \lambda \, \tau \, \|x_{-1}\|^2 \leq \beta_1(\nu, x_{-1})$$

for a suitable constant scalar τ and vector price adjustment v. In this case, essentially the same technique of proof used above for Walrasian linear budget sets could be used to give very similar results. Even if the non-Walrasian mechanism under consideration is not decentralizable by smooth convex budget sets $B(\nu)$, it might still be possible to adapt the arguments due to Trockel [34] and others in order to show that mean demand converges to a smooth function as the economy becomes large. Nevertheless, a completely different proof technique would be needed.

Such an extension could also encompass mechanisms involving taxes that are used to finance the production of a fixed bundle of public goods. Of course, except in a few special cases, we do not envisage being able to find any even asymptotically strategy-proof mechanism for determining an asymptotically efficient allocation of public goods.

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