Subjectively Expected State-Independent Utility on State-Dependent Consequence Domains

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Abstract

The standard decision theories of Savage and of Anscombe and Aumann both postulate that the domain of consequences is state independent. But this hypothesis makes no sense when, for instance, there is a risk of death or serious injury. The paper considers one possible way of deriving subjective probabilities and utilities in this case also. Moreover, the utilities will be state independent in the sense of giving equal value to any consequence that happens to occur in more than one state dependent consequence domain. The key is to consider decision trees having "hypothetical" probabilities attached to states of nature, and even to allow hypothetical choices of these probabilities.

1 Introduction: State-Dependent Consequence Domains

The standard decision theories of Savage (1954) and of Anscombe and Aumann (1963) both rely on the assumption that there are "constant acts" yielding the same consequence in all states of the world. More precisely, they postulate that the domain of consequences is state independent. But there are many decision problems where this hypothesis makes no sense — for instance, where there is a risk of death or serious injury. The point was first made by Drèze (1958, 1961) that such problems do not fit well with Savage's (1954) assumption that all consequences are possible in every state of the world.

The inapplicability of standard theory led several authors to investigate state-dependent utilities — see especially Karni (1985, 1987), Drèze (1987b), and the works cited therein, together with Jones-Lee (1979). Obviously, state-dependent utility is a generalization of the standard theory. Yet a more satisfactory generalization would reduce to the standard theory with state-independent utility in the special case considered by that theory — namely, when there is a state-independent consequence domain. In particular, the von Neumann–Morgenstern utility function (NMUF) should be state-independent in the sense of giving a unique value to any consequence that happens to occur in more than one state-dependent consequence domain. Such a generalization was provided by Fishburn (1970, Section 13.2) for a special case when there at least two non-indifferent common consequences belonging to each state-dependent consequence domain. This paper sets out to provide a similar generalization for general state-dependent consequence domains.

In the rest of the paper, Section 2 begins by reviewing evaluation functions (Wilson, 1968) which are defined on pairs consisting of states of the world and consequences. It shows how they relate to marginal rates of substitution between appropriate probability shifts, and how this relationship implies that an evaluation function is determined up to a unique co-cardinal equivalence class. Next, Section 3 recalls five sufficient conditions for the existence of an evaluation function whose expected value is maximized by the agent's behaviour.[1] To allow subjective probabilities to be disentangled from the evaluation function, Section 4 analyses decision problems with "hypothetical" probabilities attached to states of nature, following the suggestion of Karni, Schmeidler and Vind (1983). It even allows hypothetical choices of these probabilities, as in Drèze (1961, 1987) and also Karni (1985). Finally, Section 5 invokes a weaker form of the standard state independence condition which is appropriate for state dependent consequence domains. It also presents the main theorem guaranteeing the existence of a state-independent NMUF even when the consequence domain depends on the state.

2 Evaluation Functions

Let S be a fixed finite domain of possible states of the world. This paper considers the implications of allowing state-dependent consequence domains Y_s ($s \in S$). Also, in contrast to Karni (1993a, b), it will not be assumed that consequences in different state-dependent consequence domains are in any way related through "constant valuation acts" or "state invariance".

The Cartesian product space $Y^S := \prod_{s \in S} Y_s$ has members $y^S = \langle y_s \rangle_{s \in S}$ in the form of mappings from states to consequences. Savage calls these "acts" whereas Anscombe and Aumann refer to "horse lotteries," but I prefer to call them *contingent consequence functions* (or CCFs). Following Anscombe and Aumann, consider the space $\Delta(Y^S)$ of simple ("roulette") lotteries over Y^S. Each $\lambda^S \in \Delta(Y^S)$ specifies the (objective) probability $\lambda^S(y^S)$ of each CCF $y^S \in Y^S$; these probabilities are positive only for a finite set of different CCFs.

Eventually, Lemma 2 in Section 3 will demonstrate what happens when Anscombe and Aumann's state independence condition is dropped, but their other axioms are applied to the domain $\Delta(Y^S)$. In order to state the result, first define the *union domain* $\hat{Y} := \cup_{s \in S} Y_s$ of all consequences that can occur in some state of the world. Then let

$$Y_S := \cup_{s \in S}(\{s\} \times Y_s) = \{\, (s, y) \in S \times \hat{Y} \mid y \in Y_s \,\} \tag{1}$$

be the *universal domain* of state–consequence pairs. This is an obvious generalization of the domain of "prize–state lotteries" considered by Karni (1985).

Next, for each $s \in S$ and $y \in Y_s$, define $Y^S_s(y) := \{\, y^S \in Y^S \mid y_s = y \,\}$ as the set of CCFs yielding the particular consequence y in state s. Then the *marginal*

[1]See also Myerson (1979) for a somewhat different treatment of this issue.

probability of consequence y in state s is given by

$$\lambda_s(y) := \sum\nolimits_{y^S \in Y_s^S(y)} \lambda^S(y^S) \tag{2}$$

These probabilities specify the *marginal distribution* $\lambda_s \in \Delta(Y_s)$ on the appropriate component Y_s of the product space Y^S.

Throughout this paper it will be assumed that there is a (complete and transitive) preference ordering $\succsim$ on $\Delta(Y^S)$. Given this ordering, define an *evaluation function* (Wilson, 1968; Myerson, 1979) as a real-valued mapping $w(s, y)$ on the domain Y_S with the property that the preference ordering $\succsim$ is represented by the expected total evaluation defined for all $\lambda^S \in \Delta(Y^S)$ by

$$U^S(\lambda^S) = \sum\nolimits_{y^S \in Y^S} \lambda^S(y^S) \sum\nolimits_{s \in S} w(s, y_s) = \sum\nolimits_{s \in S} \sum\nolimits_{y_s \in Y_s} \lambda_s(y_s)\, w(s, y_s) \tag{3}$$

Note that evaluation functions differ from state-dependent utility functions because the latter are separate from subjective probabilities, whereas the former combine utility functions with subjective probabilities. Note too how (3) implies that only the marginal probabilities $\lambda_s(y)$ $(s \in S,\ y \in Y)$ are relevant to the expected evaluation.

Say that two evaluation functions $w(s, y)$ and $\tilde{w}(s, y)$ are *co-cardinally equivalent* if and only if there exist real constants $\rho > 0$, independent of s, and δ_s $(s \in S)$, such that

$$\tilde{w}(s, y) = \delta_s + \rho\, w(s, y) \tag{4}$$

In this case the alternative expected evaluation satisfies

$$\tilde{U}^S(\lambda^S) = \sum\nolimits_{s \in S} \sum\nolimits_{y_s \in Y_s} \lambda_s(y_s)\, \tilde{w}(s, y_s) = \sum\nolimits_{s \in S} \delta_s + \rho\, U^S(\lambda^S) \tag{5}$$

because $\sum_{y_s \in Y_s} \lambda_s(y_s) = 1$ for each $s \in S$. Hence $\tilde{U}^S$ and U^S are cardinally equivalent, so both represent the same preference ordering on $\Delta(Y^S)$.

Conversely, suppose that (3) and (5) both represent the same ordering $\succsim$ on $\Delta(Y^S)$. Let s, s' be any pair of states in S, and $a, b \in Y_s$, $c, d \in Y_{s'}$ any four consequences with $w(s, a) \neq w(s, b)$ and $w(s', c) \neq w(s', d)$. Consider any shift Δ_s in probability from consequence b to a in state s, and also any shift $\Delta_{s'}$ in probability from consequence d to c in state s'. If preferences are represented by (3), such shifts leave the agent indifferent if and only if

$$[w(s, a) - w(s, b)]\, \Delta_s + [w(s', c) - w(s', d)]\, \Delta_{s'} = 0$$

Similarly, if preferences are represented by (5). Hence, the common ratio

$$\frac{w(s, a) - w(s, b)}{w(s', c) - w(s', d)} = \frac{\tilde{w}(s, a) - \tilde{w}(s, b)}{\tilde{w}(s', c) - \tilde{w}(s', d)} \tag{6}$$

of evaluation differences is equal to the constant marginal rate of substitution (MRS) between shifts Δ_s and $\Delta_{s'}$.[2] So for all such configurations of s, s', a, b, c, d

[2]This extends an idea due to Machina (1987, pp. 125–6).

there must exist some constant $\rho > 0$ such that

$$\frac{\tilde{w}(s,a) - \tilde{w}(s,b)}{w(s,a) - w(s,b)} = \frac{\tilde{w}(s',c) - \tilde{w}(s',d)}{w(s',c) - w(s',d)} = \rho$$

This implies (4), so $\tilde{w}(s,y)$ and $w(s,y)$ must be co-cardinally equivalent functions on the domain Y_S.

3 Five Sufficient Conditions

Anscombe and Aumann postulated that the expected utility hypothesis was satisfied for lotteries with objective probabiliities. When applied to lotteries in $\Delta(Y^S)$, this hypothesis implies and implied by the following three conditions:

(O) *Ordering.* There exists a (compete and transitive) preference ordering $\succsim$ on $\Delta(Y^S)$.

(I*) *Strong Independence Axiom.* For any $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S)$ and $0 < \alpha \leq 1$, it must be true that

$$\lambda^S \succsim \mu^S \iff \alpha\lambda^S + (1-\alpha)\nu^S \succsim \alpha\mu^S + (1-\alpha)\nu^S$$

(C*) *Strong Continuity as Probabilities Vary.* For each $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S)$ with $\lambda^S \succ \mu^S \succ \nu^S$, the two sets

$$A := \{\alpha \in [0,1] \mid \alpha\lambda^S + (1-\alpha)\nu^S \succsim \mu^S\}$$
$$B := \{\alpha \in [0,1] \mid \alpha\lambda^S + (1-\alpha)\nu^S \precsim \mu^S\}$$

must both be closed in $[0,1]$.

As shown by Jensen (1967) and Fishburn (1970), the expected utility hypothesis is still implied when conditions (I*) and (C*) are replaced by the following two weaker conditions, both of which apply for each $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S)$:

(I) *Independence.* Whenever $0 < \alpha \leq 1$, then

$$\lambda^S \succ \mu^S \Longrightarrow \alpha\lambda^S + (1-\alpha)\nu^S \succ \alpha\mu^S + (1-\alpha)\nu^S$$

(C) *Continuity.* Whenever $\lambda^S \succ \mu^S \succ \nu^S$, there must exist $\alpha', \alpha'' \in (0,1)$ such that

$$\alpha'\lambda^S + (1-\alpha')\nu^S \succ \mu^S \quad \text{and} \quad \mu^S \succ \alpha''\lambda^S + (1-\alpha'')\nu^S$$

As already discussed, one implication of (3) is the following condition:

(RO) *Reversal of Order.* Whenever $\lambda^S, \mu^S \in \Delta(Y^S)$ have marginal distributions satisfying $\lambda_s = \mu_s$ for all $s \in S$, then $\lambda^S \sim \mu^S$.

This condition owes its name to the fact that there is indifference between: (i) the compound lottery in which a roulette lottery λ^S determines the random CCF y^S before the horse lottery that resolves which state $s \in S$ and which ultimate consequence y_s occur; and (ii) the reversed compound lottery in which the horse lottery is resolved first, and its outcome $s \in S$ determines which marginal roulette lottery λ_s generates the ultimate consequence y.

In particular, suppose that $\mu^S = \prod_{s \in E} \lambda_s$ is the *product lottery* defined, for all $y^S = \langle y_s \rangle_{s \in S} \in Y^S$, by $\mu^S(y^S) := \prod_{s \in S} \lambda_s(y_s)$. Thus, the different random consequences y_s ($s \in S$) all have independent distributions. Then condition (RO) requires λ^S to be treated as equivalent to μ^S, whether or not the different elementary consequences y_s ($s \in S$) are correlated random variables when the joint distribution is λ^S. Only marginal distributions matter. So any $\lambda^S \in \Delta(Y^S)$ can be regarded as equivalent to the list $\langle \lambda_s \rangle_{s \in S}$ of corresponding marginal distributions. This has the effect of reducing the space $\Delta(Y^S)$ to the Cartesian product space $\prod_{s \in S} \Delta(Y_s)$.

An *event* is any non-empty subset E of S. For each event $E \subset S$, let Y^E denote the corresponding Cartesian subproduct $\prod_{s \in E} Y_s$, and let $\Delta(Y^E)$ denote the space of lotteries λ^E, μ^E, ν^E, etc. with outcomes $y^E \in Y^E$. Then (3) implies that there is a corresponding *contingent expected utility function*

$$U^E(\lambda^E) = \sum\nolimits_{s \in E} \sum\nolimits_{y_s \in Y_s} \lambda_s(y_s)\, w(s, y_s) \tag{7}$$

which represents the *contingent preference ordering* $\succsim^E$ on the set $\Delta(Y^E)$.

Suppose that $\lambda^E, \mu^E \in \Delta(Y^E)$ and $\nu^{S \setminus E} \in \Delta(Y^{S \setminus E})$. Let $(\lambda^E, \nu^{S \setminus E})$ denote the combination of the conditional lottery λ^E if E occurs with $\nu^{S \setminus E}$ if $S \setminus E$ occurs, and similarly for $(\mu^E, \nu^{S \setminus E})$. Note that when $\succsim$ is represented by $U^S(\lambda^S)$ defined by (3), then

$$\begin{aligned}
\lambda^E \ \succsim^E \ \mu^E \quad &\iff \quad U^E(\lambda^E) \geq U^E(\mu^E) \\
&\iff \quad \sum\nolimits_{s \in E} \sum\nolimits_{y_s \in Y_s} [\lambda_s(y_s) - \mu_s(y_s)]\, w(s, y_s) \\
&\iff \quad U^S(\lambda^E, \nu^{S \setminus E}) \geq U^S(\mu^E, \nu^{S \setminus E}) \\
&\iff \quad (\lambda^E, \nu^{S \setminus E}) \ \succsim \ (\mu^E, \nu^{S \setminus E})
\end{aligned}$$

So the following version of the usual sure thing principle must hold:

(STP) *Sure Thing Principle.* Given any event $E \subset S$, there exists a contingent preference ordering $\succsim^E$ on $\Delta(Y^E)$ satisfying

$$\lambda^E \ \succsim^E \ \mu^E \iff (\lambda^E, \nu^{S \setminus E}) \ \succsim \ (\mu^E, \nu^{S \setminus E}) \tag{8}$$

for all $\lambda^E, \mu^E \in \Delta(Y^E)$ and all $\nu^{S \setminus E} \in \Delta(Y^{S \setminus E})$.

The following preliminary Lemma 1 shows that the four conditions (O), (I*), (RO) and (STP) are not logically independent. In fact, as Raiffa (1961) implicitly suggests in his discussion of the Ellsberg paradox, condition (STP) is an implication of the three conditions (O), (I*) and (RO) — see also Blume, Brandenburger and Dekel (1991).

Lemma 1. *Suppose that the three axioms (O), (I*), and (RO) are satisfied on* $\Delta(Y^S)$. *Then so is (STP).*

Proof: Consider any event $E \subset S$ and also any lotteries $\lambda^E, \mu^E \in \Delta(Y^E)$, $\bar{\nu}^{S\setminus E} \in \Delta(Y^{S\setminus E})$ satisfying $(\lambda^E, \bar{\nu}^{S\setminus E}) \precsim (\mu^E, \bar{\nu}^{S\setminus E})$. For any other lottery $\nu^{S\setminus E} \in \Delta(Y^{S\setminus E})$, axioms (I*) and (RO) respectively imply that

$$\begin{aligned} \tfrac{1}{2}(\lambda^E, \nu^{S\setminus E}) + \tfrac{1}{2}(\lambda^E, \bar{\nu}^{S\setminus E}) &\precsim \tfrac{1}{2}(\lambda^E, \nu^{S\setminus E}) + \tfrac{1}{2}(\mu^E, \bar{\nu}^{S\setminus E}) \\ &\sim \tfrac{1}{2}(\mu^E, \nu^{S\setminus E}) + \tfrac{1}{2}(\lambda^E, \bar{\nu}^{S\setminus E}) \end{aligned}$$

But then transitivity of $\precsim$ and axiom (I*) imply that $(\lambda^E, \nu^{S\setminus E}) \precsim (\mu^E, \nu^{S\setminus E})$. This confirms that one can use (8) to define the contingent preference relation $\precsim^E$ on $\Delta(Y^E)$. So condition (STP) is satisfied. ∎

The next result confirms that the five conditions presented so far are sufficient for the existence of an evaluation function. Beforehand, however, assume that for every state $s \in S$, there exist $\bar{\lambda}_s, \underline{\lambda}_s \in \Delta(Y_s)$ such that the contingent ordering $\precsim^{\{s\}}$ on $\Delta(Y_s)$ satisfies $\bar{\lambda}_s \succ^{\{s\}} \underline{\lambda}_s$. This assumption really loses no generality because, by (STP), states without this property can be omitted from S without affecting preferences over random CCFs for the remaining states. In fact, it is like removing all null states in Savage's theory. Obviously, $\bar{\lambda}^S \succ \underline{\lambda}^S$, as can be shown by repeated application of condition (STP).

Lemma 2. *Under the five conditions (O), (I), (C), (RO) and (STP), there exists a unique co-cardinal equivalence class of evaluation functions* $w(s, y)$ *such that the expected sum* $U^S(\lambda^S)$ *defined by (3) represents the corresponding preference ordering* $\precsim$ *on* $\Delta(Y^S)$.

Proof: Because the ordering $\precsim$ satisfies conditions (O), (I) and (C), a standard result of (objectively) expected utility theory shows that $\precsim$ can be represented by a unique normalized expected utility function $U^S : \Delta(Y^S) \to \mathbb{R}$ which satisfies the equations

$$U^S(\underline{\lambda}^S) = 0 \quad \text{and} \quad U^S(\bar{\lambda}^S) = 1 \tag{9}$$

as well as the *mixture preservation* property (MP) requiring that, whenever $\lambda^S, \mu^S \in \Delta(Y^S)$ and $0 \le \alpha \le 1$, then

$$U^S(\alpha\, \lambda^S + (1-\alpha)\, \mu^S) = \alpha\, U^S(\lambda^S) + (1-\alpha)\, U^S(\mu^S) \tag{10}$$

Then for each state $s \in S$ and lottery $\lambda \in \Delta(Y_s)$, define

$$u_s(\lambda) := U^S(\underline{\lambda}^{S\setminus\{s\}}, \lambda) \tag{11}$$

Let m be the number of elements in the finite set S. By an argument similar to that used by Fishburn (1970), for all $\lambda^S \in \Delta(Y^S)$, condition (RO) implies that the two members

$$\sum\nolimits_{s\in S} \frac{1}{m}\, (\underline{\lambda}^{S\setminus\{s\}}, \lambda_s) \quad \text{and} \quad \frac{m-1}{m}\, \underline{\lambda}^S + \frac{1}{m}\, \lambda^S \tag{12}$$

of $\Delta(Y^S)$ are indifferent because for each $s \in S$ they have the common marginal distribution $(1 - \frac{1}{m})\,\underline{\lambda}_s + \frac{1}{m}\,\lambda_s$. Because U^S satisfies (MP), applying U^S to the two indifferent mixtures in (12) gives the equality

$$\sum\nolimits_{s\in S} \frac{1}{m}\,U^S(\underline{\lambda}^{S\setminus\{s\}}, \lambda_s) = \frac{m-1}{m}\,U^S(\underline{\lambda}^S) + \frac{1}{m}\,U^S(\lambda^S) \tag{13}$$

But $U^S(\underline{\lambda}^S) = 0$ by (9), so (11) and (13) imply that

$$U^S(\lambda^S) = \sum\nolimits_{s\in S} u_s(\lambda_s) \tag{14}$$

Finally, for each $y \in Y_s$, let $1_y \in \Delta(Y_s)$ denote the degenerate lottery attaching probability 1 to the particular consequence y. Then define $w(s, y) := u_s(1_y)$ for each $s \in S$ and $y \in Y_s$. By (11), because U^S satisfies (10), one has

$$\begin{aligned} u_s(\alpha\,\lambda_s + (1-\alpha)\,\mu_s) &= U^S(\underline{\lambda}^{S\setminus\{s\}}, \alpha\,\lambda_s + (1-\alpha)\,\mu_s) \\ &= \alpha\,U^S(\underline{\lambda}^{S\setminus\{s\}}, \lambda_s) + (1-\alpha)\,U^S(\underline{\lambda}^{S\setminus\{s\}}, \mu_s) \\ &= \alpha\,u_s(\lambda_s) + (1-\alpha)\,u_s(\mu_s) \end{aligned}$$

whenever $\lambda_s, \mu_s \in \Delta(Y_s)$ and $0 \le \alpha \le 1$. Hence, u_s also satisfies an appropriate version of (MP) and so, because $\lambda_s \equiv \sum_{y\in Y_s} \lambda_s(y)\,1_y$, it follows that $u_s(\lambda_s) \equiv \sum_{y\in Y_s} \lambda_s(y)\,w(s, y)$. Because of (14), $U^S(\lambda^S)$ is given by (3).

The fact that there is a unique co-cardinal equivalence class of the functions $w(s, y)$ follows easily from the discussion at the end of Section 2. ∎

4 Chosen Probabilities and State-Dependent Utilities

An extreme case of state-dependent consequence domains occurs if Y_s and $Y_{s'}$ are disjoint whenever $s \neq s'$. In this case, there is no hope of inferring subjective probabilities from behaviour. To see why, suppose that the agent's behaviour is observed to maximize the subjective expected utility (SEU) function

$$U^S(\lambda^S) = \sum\nolimits_{s\in S} p_s \sum\nolimits_{y_s\in Y_s} \lambda_s(y_s)\,v(y_s)$$

where $p_s > 0$ for all $s \in S$. Then the same behaviour will also maximize the equivalent SEU function

$$U^S(\lambda^S) = \sum\nolimits_{s\in S} \tilde{p}_s \sum\nolimits_{y_s\in Y_s} \lambda_s(y_s)\,\tilde{v}(y_s)$$

for *any* positive subjective probabilities $\tilde{p}_s$ satisfying $\sum_{s\in S} \tilde{p}_s = 1$, provided that $\tilde{v}(y) = p_s\,v(y_s)/\tilde{p}_s$ for all $y \in Y_s$. Without further information, there is no way of disentangling subjective probabilities from utilities.

Following a suggestion of Karni, Schmeidler and Vind (1983), such additional information could be inferred from hypothetical behaviour when probabilities p_s ($s \in S$) happen to be specified. The idea is that, though the agent does not know the true probabilities of the different states of the world, nevertheless it should be possible for coherent decisions to emerge if the agent happened to discover what the true probabilities are. In particular, if the true probabilities happen to coincide with the agent's subjective probabilities, the agent's behaviour should be the same whether or not these true probabilities are known.[3]

A somewhat extreme version of this assumption will be used here. Following Karni (1985, Section 1.6), Schervish, Seidenfeld and Kadane (1990), and also Karni and Schmeidler (1991), it will be assumed that the decision-maker can handle problems involving not only hypothetical probabilities, but also hypothetical choices of probabilities. As discussed by Karni and Mongin (1997), these hypothetical choices involve what they call "state–outcome lotteries". Consider, for instance, problems where the states of nature are indeed natural disasters, weather events, etc. It will be assumed that the decision-maker can rank prospects of the following general kind: A probability of 2% each year of a major earthquake? Or 1% each year of a devastating hundred-year flood? Or 4% each year of a serious forest fire set off by lightning? More specifically, the assumption is that the decision-maker can resolve such issues within a coherent framework of decision analysis. Certainly, if the SEU hypothesis holds, it can be applied to decide such issues. Drèze's (1961, 1987) theory of "moral hazard" is based on a somewhat related idea. But Drèze assumes that the agent can influence the choice of state, as opposed to the choice of probabilities of different states.

For this reason, it will be assumed that there exists an additional preference ordering $\succsim_S$ on the whole extended lottery domain $\Delta(Y_S)$, where Y_S is defined by (1) — i.e., it is the universal state–consequence domain of pairs (s, y). Thus, $\succsim_S$ satisfies condition (O). Furthermore, assume that $\succsim_S$ satisfies the obvious counterparts of conditions (I) and (C) for the domain $\Delta(Y_S)$.[4] Arguing as in the orthodox theory of (objectively) expected utility, there must exist a unique cardinal equivalence class of extended NMUFs v_S on the domain Y_S whose expected values all represent the ordering $\succsim_S$ on $\Delta(Y_S)$. Because the function $v_S(s, y)$ has both the state $s \in S$ and the consequence $y \in Y_s$ as arguments, for each fixed $s \in S$ the NMUF $v_S(s, \cdot)$ is a state-dependent utility function on the domain Y_s.

Note next that when any state $s \in S$ is certain, and assuming that everything relevant to each decision is included within each consequence $y \in Y_s$, the spaces Y_s and $Y_{Ss} := \{s\} \times Y_s$ are effectively equivalent consequence domains. Thus,

[3] Recently Mongin (1997), then Karni and Mongin (1997) have pointed out a serious defect with the approach due to Karni, Schmeidler and Vind. The problem is that alternative specifications of the "hypothetical" probabilities p_s $(s \in S)$ can easily lead to different subjective probabilities, in general.

[4] These extended versions of conditions (O) and (I) can be given a consequentialist justification along the lines of Hammond (1988). This is done by considering a suitably extended domain of decision trees in which natural nodes become replaced by chance nodes, and there are even several copies of natural nodes so that opportunities to affect the probabilities attached to states of nature are incorporated in the tree.

each $\Delta(Y_s)$ is effectively the same space as the set

$$\Delta(Y_{Ss}) := \{ \lambda \in \Delta(Y_S) \mid \lambda(\{s\} \times Y_s) = 1 \} \tag{15}$$

of lotteries attaching probability one to the state $s \in S$. So it will be assumed that the contingent preference ordering $\succsim^{\{s\}}$ on $\Delta(Y_s)$ is identical to the ordering $\succsim_S$ restricted to $\Delta(Y_{Ss})$. But these orderings are represented by the expected values of the two respective NMUFs $w(s,y)$ and $v_S(s,y)$ on the common domain Y_s. So these NMUFs are cardinally equivalent. Hence, there must exist constants $\rho_s > 0$ and δ_s such that on Y_s one has

$$w(s,y) \equiv \delta_s + \rho_s \, v_S(s,y) \tag{16}$$

Let $\rho := \sum_{s \in S} \rho_s$. Obviously $\rho > 0$.

Next, define the ratios $q_s := \rho_s/\rho$ for all $s \in S$. Clearly each $q_s > 0$ and $\sum_{s \in S} q_s = 1$. Therefore the ratios q_s can be interpreted as subjective probabilities. Furthermore, $\succsim$ on $\Delta(Y^S)$ is represented by the expectation of the NMUF $v^S(y^S) := \sum_{s \in S} q_s \, v_S(s, y_s)$.

Given the CCF $y^S \in Y^S$ and consequence $y \in \hat{Y} = \cup_{s \in S} Y_s$, let

$$E(y^S, y) := \{ s \in S \mid y_s = y \}$$

be the set of states in which y occurs. Then the CCF $y^S \in Y^S$ is subjectively equivalent to the lottery $\lambda \in \Delta(\hat{Y})$ with the objective probability of each consequence $y \in \hat{Y}$ given by $\lambda(y) = \sum_{s \in E(y^S, y)} q_s$.

Because of (16), one has $w(s, \tilde{y}_s) - w(s, y_s) = \rho_s \, [v_S(s, \tilde{y}_s) - v_S(s, y_s)]$ for any state $s \in S$ and any pair of consequences $y_s, \tilde{y}_s \in Y_s$. Therefore,

$$\frac{q_s}{q_{s'}} = \frac{\rho_s}{\rho_{s'}} = \frac{w(s, \tilde{y}_s) - w(s, y_s)}{w(s', \tilde{y}_{s'}) - w(s', y_{s'})} \cdot \frac{v_S(s', \tilde{y}_{s'}) - v_S(s', y_{s'})}{v_S(s, \tilde{y}_s) - v_S(s, y_s)} \tag{17}$$

This formula now enables ratios of subjective probabilities to be inferred uniquely in an obvious way from marginal rates of substitution (MRSs) between shifts in objective probability, expressed in the form of ratios of utility differences. The first term of the product is the MRS between changes in the probabilities of consequences in two different states of the kind considered in (6). The second term is a four-way ratio of utility differences that equals the MRS between shifts in probability from $(s', \tilde{y}_{s'})$ to $(s', y_{s'})$ and shifts in probability from $(s, \tilde{y}_s)$ to (s, y_s). One particular advantage of Anscombe and Aumann's approach is that subjective probabilities can be interpreted in this way. No interpretation quite as simple emerges from Savage's version of the theory.

To summarize the results of the above discussion:

Lemma 3. *Suppose that:*

1. *conditions (O), (I), and (C) apply to the ordering $\succsim_S$ on the domain $\Delta(Y_S)$;*

2. *conditions (O), (I), (C), (RO) and (STP) apply to the ordering $\succsim$ on $\Delta(Y^S)$;*

3. *for each $s \in S$, the contingent preference ordering $\succsim^{\{s\}}$ on $\Delta(Y_s)$ is identical to the restriction of the ordering $\succsim_S$ to this set, regarded as equal to $\Delta(Y_{Ss})$ defined by (15);*

4. *for each $s \in S$, there exist $\bar{\lambda}_s, \underline{\lambda}_s \in \Delta(Y_s)$ such that $\bar{\lambda}_s \succ^{\{s\}} \underline{\lambda}_s$.*

Then there exist unique positive subjective probabilities q_s ($s \in S$) and a unique cardinal equivalence class of state-dependent NMUFs $v_S : Y_S \to \mathbb{R}$ such that the ordering $\succsim$ on $\Delta(Y^S)$ is represented by the expected utility function

$$U^S(\lambda^S) \equiv \sum_{s \in S} q_s \sum_{y_s \in Y_s} \lambda_s(y_s)\, v_S(s, y_s) \tag{18}$$

5 State-Independent Utilities

Previous writers have expressed a specific interest in state-dependent preferences and utilities. There was no attempt to define the space of consequences broadly enough so that the preference between any pair of (risky) consequences would be independent of the state in which they both occur. This flies in the face of the traditional approach to decision theory, in which actions are valued entirely by their consequences. It also contradicts the closely related "consequentialist" approach, which recommends that behaviour all decision trees should effectively reveal a consequence choice function (Hammond, 1988).

The motivation which Karni, Schmeidler and Vind (1983) in particular offer for state-dependent preferences is to treat "a class of insurance problems involving irreplaceable objects such as life, health and heirlooms," and also "criminal activity where one possible outcome is loss of freedom" (p. 1021). These writers infer that "[t]here are circumstances ... in which the evaluation of the consequences is not independent of the prevailing state of nature". No doubt this is true if one insists on considering only (narrow) economic consequences such as commodity bundles or purchasing power. But if life, health, heirlooms, and freedom are really relevant to good decisions, I would argue that they should be included in the descriptions of consequences.

In fact, no attention has been paid so far to the evident fact that some consequences can arise in more than one state of the world. Apart from being unrealistic, this also means that the usual theory of subjective expected utility has not really been generalized. Instead of one extreme of identical consequence domains in all states, as in the classical theory, most of the existing literature has merely gone to the other extreme of consequence domains in different states being treated as if they were pairwise disjoint. The main point of this paper is to find sufficient conditions for giving a unique value to each consequence, even if it occurs in a different state of the world.

So there is no good case for requiring the value of a consequence to depend upon the state of the world in which it occurs — as Arrow (1974, pp. 5–6) certainly recognizes, for one. Drèze (1987a, ch. 2) also discusses this point, but prefers a theory of preferences regarding "prizes" (such as "money amounts or commodity bundles") which can be associated with every state, so that conditional preferences on a fixed set of prizes are well defined for every possible event. Drèze (1987a, p. 28) is fully aware that this "amounts to redefining consequences as pairs, consisting of a prize and a state".

Actually, what appear to be "state-dependent" preferences for prizes in state–prize pairs are trivially equivalent to state-independent preferences for suitably defined extended consequences. To see this, suppose one regards each state–outcome pair (s, y) in the universal domain Y_S defined by (1) as a unique extended consequence in a space $\tilde{Y}$ equal to Y_S. Then the union domain $\hat{Y} = \cup_{s \in S} Y_s$ satisfies $\hat{Y} = \tilde{Y} = Y_S$, and the state-dependent NMUF $v_S(s, y)$ of Lemma 3 is equivalent to a unique state-independent NMUF $\tilde{v} : \tilde{Y} \to \mathbb{R}$.

Reverting now to the general argument, let us note first that there is a natural embedding $\phi : \Delta(Y_S) \to \Delta(\hat{Y})$ from lotteries over the universal domain Y_S of state–consequence pairs to lotteries over the union consequence domain $\hat{Y} = \cup_{s \in S} Y_s$. After adopting the convention that $\lambda_S(s, y) = 0$ whenever $y \notin Y_s$, this embedding can be defined by

$$\phi(\lambda_S)(y) := \sum\nolimits_{s \in S} \lambda_S(s, y) \tag{19}$$

for all $\lambda_S \in \Delta(Y_S)$ and all $y \in \hat{Y}$. Thus, $\phi(\lambda_S)(y)$ is the total probability of all state–consequence pairs (s, y) in which the particular consequence y occurs. Evidently, for all $\lambda_S, \mu_S \in \Delta(Y_S)$ and all $\alpha \in (0, 1)$, definition (19) implies that

$$\phi(\alpha\, \lambda_S + (1 - \alpha)\, \mu_S) = \alpha\, \phi(\lambda_S) + (1 - \alpha)\, \phi(\mu_S) \tag{20}$$

Lemma 4. *The mapping $\phi : \Delta(Y_S) \to \Delta(\hat{Y})$ is onto.*

Proof: Given any $\lambda \in \Delta(\hat{Y})$, let $K_\lambda := \{\, y \in \hat{Y} \mid \lambda(y) > 0 \,\}$ denote the support of the distribution λ. For each consequence $y \in K_\lambda$, choose any state $s(y) \in S$ with the property that $y \in Y_{s(y)}$; at least one such state always exists. Then define $\lambda_S \in \Delta(Y_S)$ so that $\lambda_S(s(y), y) = \lambda(y)$ for all $y \in K_\lambda$, but $\lambda_S(s, y) = 0$ unless both $y \in K_\lambda$ and $s = s(y)$. Evidently $\phi(\lambda_S)(y) = \lambda_S(s(y), y) = \lambda(y)$ for all $y \in K_\lambda$, and $\phi(\lambda_S)(y) = \lambda(y) = 0$ for all $y \notin K_\lambda$. ∎

The pre-image correspondence $\Phi_S : \Delta(\hat{Y}) \twoheadrightarrow \Delta(Y_S)$ of ϕ can be defined, for all $\lambda \in \Delta(\hat{Y})$, by

$$\Phi_S(\lambda) := \{\, \lambda_S \in \Delta(Y_S) \mid \phi(\lambda_S) = \lambda \,\} \tag{21}$$

Because of Lemma 4, $\Phi_S(\lambda)$ is never empty. In this framework, it now seems natural to impose the requirement that, given any pair $\lambda_S, \mu_S \in \Delta(Y_S)$ for which the induced consequence lotteries $\phi(\lambda_S), \phi(\mu_S) \in \Delta(\hat{Y})$ are the same, the state s in which each state–consequence pair $(s, y) \in Y_S$ occurs is irrelevant. In particular, this suggests the following:

(GSI) *Generalized State Independence.* For all pairs $\lambda_S, \mu_S \in \Delta(Y_S)$ one has $\lambda_S \sim_S \mu_S$ whenever $\phi(\lambda_S) = \phi(\mu_S)$.

Thus, for each $\lambda \in \Delta(\hat{Y})$, the set $\Phi_S(\lambda)$ must be an indifference class for the relation $\succsim_S$. So there must exist a "state-independent consequence" preference relation $\succsim_Y$ on $\Delta(\hat{Y})$ defined by

$$\lambda \succsim_Y \mu \iff [\forall \lambda_S \in \Phi_S(\lambda); \forall \mu_S \in \Phi_S(\mu) : \lambda_S \succ_S \mu_S] \tag{22}$$

Equivalently, for all pairs $\lambda_S, \mu_S \in \Delta(Y_S)$, it must be true that

$$\lambda_S \succsim_S \mu_S \iff \phi(\lambda_S) \succsim_Y \phi(\mu_S)$$

In the special case of a state-independent consequence domain, when $Y_s = Y$ for all $s \in S$, condition (GSI) evidently implies that $\succsim_S$ reduces to an ordering on $\Delta(Y)$. But condition (GSI) can also hold when the domains Y_s depend on the state; they could even be pairwise disjoint.

Lemma 5. *Suppose that conditions (O), (I), (C) and (GSI) apply to the ordering $\succsim_S$ on the domain $\Delta(Y_S)$. Then the relation $\succsim_Y$ on $\Delta(\hat{Y})$ defined by (22) satisfies conditions (O), (I), and (C).*

PROOF: Throughout the following proof, given any three lotteries $\lambda, \mu, \nu \in \Delta(\hat{Y})$, let $\lambda_S, \mu_S, \nu_S \in \Delta(Y_S)$ denote arbitrarily chosen members of $\Phi_S(\lambda)$, $\Phi_S(\mu)$ and $\Phi_S(\nu)$ respectively. That is, suppose $\lambda = \phi(\lambda_S)$, $\mu = \phi(\mu_S)$, and $\nu = \phi(\nu_S)$. Because of (20), whenever $0 \leq \alpha \leq 1$ it follows that

$$\begin{aligned} \phi(\alpha\, \lambda_S + (1-\alpha)\, \nu_S) &= \alpha\, \lambda + (1-\alpha)\, \nu \\ \text{and} \quad \phi(\alpha\, \mu_S + (1-\alpha)\, \nu_S) &= \alpha\, \mu + (1-\alpha)\, \nu \end{aligned} \tag{23}$$

Condition (O). Because (GSI) implies that each set $\Phi_S(\lambda)$ $(\lambda \in \Delta(\hat{Y}))$ must be an indifference class for the preference ordering $\succsim_S$, definition (22) obviously implies that $\succsim_Y$ is reflexive, complete, and transitive. So $\succsim_Y$ is a preference ordering.

Condition (I). Suppose that $0 < \alpha < 1$. Because $\succsim_S$ satisfies condition (I), it follows from (22) and (23) that

$$\begin{aligned} \lambda \succ_Y \mu \quad \iff \quad \lambda_S \succ_S \mu_S \quad &\implies \quad \alpha\, \lambda_S + (1-\alpha)\, \nu_S \succ_S \alpha\, \mu_S + (1-\alpha)\, \nu_S \\ &\iff \quad \alpha\, \lambda + (1-\alpha)\, \nu \succ_Y \alpha\, \mu + (1-\alpha)\, \nu \end{aligned}$$

Therefore $\succsim_Y$ also satisfies condition (I).

Condition (C). Suppose that $\lambda \succ_Y \mu$ and $\mu \succ_Y \nu$. Then $\lambda_S \succ_S \mu_S$ and also $\mu_S \succ_S \nu_S$. Because $\succsim_S$ satisfies condition (C), it follows that there exist $\alpha', \alpha'' \in (0,1)$ such that $\alpha'\, \lambda_S + (1-\alpha')\, \nu_S \succ_S \mu_S$ and $\mu_S \succ_S \alpha''\, \lambda_S + (1-\alpha'')\, \nu_S$. Then (20) and (23) together imply that $\alpha'\, \lambda + (1-\alpha')\, \nu \succ_Y \mu$, and also that $\mu \succ_Y \alpha''\, \lambda + (1-\alpha'')\, \nu$. Therefore $\succsim_Y$ also satisfies condition (C). ∎

Main Theorem. *Suppose that:*

1. *conditions (O), (I), (C) and (GSI) apply to the ordering $\succsim_S$ on the domain $\Delta(Y_S)$ of lotteries over state–consequence pairs (s, y) with $s \in S$ and $y \in Y_s$;*

2. *conditions (O), (I), (C), (RO) and (STP) apply to the ordering* $\succsim$ *on the domain* $\Delta(Y^S)$ *of lotteries over CCFs in the Cartesian product space* $Y^S := \prod_{s\in S} Y_s$*;*

3. *for each* $s \in S$*, the contingent preference ordering* $\succsim^{\{s\}}$ *on* $\Delta(Y_s)$ *is identical to the restriction of the ordering* $\succsim_S$ *to this set, regarded as equal to* $\Delta(Y_{Ss})$*, with* $Y_{Ss} := \{s\} \times Y_s$ *as in (15);*

4. *for each* $s \in S$*, there exist lotteries* $\underline{\lambda}_s, \overline{\lambda}_s \in \Delta(Y_s)$ *such that* $\overline{\lambda}_s \succ^{\{s\}} \underline{\lambda}_s$*.*

Then there exists a unique cardinal equivalence class of state independent NMUFs $\hat{v}$ *defined on the union consequence domain* $\hat{Y} := \cup_{s\in S} Y_s$*, as well as unique positive subjective probabilities* p_s $(s \in S)$ *such that, for every* $\hat{v}$ *in the equivalence class, the ordering* $\succsim$ *on* $\Delta(Y^S)$ *is represented by the expected value of*

$$v^S(y^S) \equiv \sum_{s\in S} p_s\, \hat{v}(y_s) \tag{24}$$

Proof: By the first hypothesis and Lemma 5, there is an associated ordering $\succsim_Y$ on $\Delta(\hat{Y})$ which satisfies conditions (O), (I), and (C). So the standard results of (objectively) expected utility theory imply that there exists a unique cardinal equivalence class of expected utility functions $\hat{U} : \Delta(\hat{Y}) \to \mathbb{R}$ which represent $\succsim_Y$ while satisfying the mixture preservation property (MP) requiring that

$$\hat{U}(\alpha\,\lambda + (1-\alpha)\,\mu) = \alpha\,\hat{U}(\lambda) + (1-\alpha)\,\hat{U}(\mu)$$

whenever $\lambda, \mu \in \Delta(\hat{Y})$ and $0 \le \alpha \le 1$.

Define $\hat{v}(y) := \hat{U}(1_y)$ for all $y \in \hat{Y}$. Then $\hat{v}$ is state-independent and belongs to a unique cardinal equivalence class. Because of (MP), condition (GSI) implies that $\succsim_S$ on $\Delta(Y_S)$ must be represented by the expected utility function

$$\begin{aligned} U_S(\lambda_S) \quad := \quad \hat{U}(\phi(\lambda_S)) \quad &= \quad \sum_{y\in Y} \phi(\lambda_S)(y)\,\hat{v}(y) \\ &= \quad \sum_{s\in S}\sum_{y\in Y_s} \lambda_S(s,y)\,\hat{v}(y) \end{aligned}$$

By the second hypothesis and Lemma 2, the ordering $\succsim$ on $\Delta(Y^S)$ is represented by the expected total evaluation given by (5) in Section 2.

Let $s \in S$ be any state. Because of the third hypothesis of the theorem, the two expected utility functions of λ_s defined by $\sum_{y\in Y_s} \lambda_s(y)\,\hat{v}(y)$ and by $\sum_{y\in Y_s} \lambda_s(y)\,w(s,y)$ must be cardinally equivalent on the domain $\Delta(Y_s)$. This implies that for each state $s \in S$, there exist constants $\rho_s > 0$ and δ_s such that $w(s,y) \equiv \delta_s + \rho_s\,\hat{v}(y)$ on Y_s. Now define $p_s := \rho_s/\rho$, where $\rho := \sum_{s\in S} \rho_s > 0$. Then each $p_s > 0$ and $\sum_{s\in S} p_s = 1$, so the constants p_s $(s \in S)$ are probabilities. Also, $w(s,y) \equiv \delta_s + \rho\,p_s\,\hat{v}(y)$. Therefore, by (3) and Lemma 2, the preference ordering $\succsim$ on $\Delta(Y^S)$ is represented by the expected value of

$$U^S(1_{y^S}) = v^S(y^S) := \sum_{s\in S} w(s,y_s) = \sum_{s\in S} \delta_s + \rho \sum_{s\in S} p_s\,\hat{v}(y_s)$$

Because $\rho > 0$, it follows that $\succsim$ is also represented by the expected value of the NMUF (24).

Finally, the subjective conditional probabilities p_s $(s \in S)$ are unique because each ratio $p_s/p_{s'}$ is given by the unique corresponding ratio (17) of utility differences. ∎

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