APPROXIMATE MEASURES OF THE SOCIAL WELFARE BENEFITS OF LARGE PROJECTS*

by

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1. Introduction

Despite the forceful criticisms which Samuelson [1947] made of Marshallian consumer surplus many years ago, it still appears to be in vogue among applied economists who have to estimate the benefits of a project. It is also to be found, usually without any criticism, in the large number of textbooks from which students learn their intermediate level microeconomics. Part of the reason for its prevalence in applied studies may be that the standard manuals on cost benefit analysis issued by the O.E.C.D. and U.N.I.D.O. (Little and Mirrless [1968] and [1974]; Dasgupta, Marglin and Sen [1972]) do not deal very successfully with the problem of evaluating large projects, for which it cannot be assumed that shadow prices are unchanged if the project is actually adopted. Yet most of the projects which governments and their advisors evaluate are large, for the very good reason that worthwhile small projects are often taken up anyway by the private sector. It is large projects which

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present problems of coordination and finance to which government intervention is most suited.

Although large projects are important, the literature in cost-benefit analysis devoted to them is extremely scanty. Negishi [1972] followed by Harris [1978] pointed out how large projects could sometimes be assessed by index number tests such as those associated with Laspeyres and Paasche. These tests are often inconclusive, however, and their validity relies on social welfare being quasi-concave as a function of the net output vector of the project, a property which can easily fail as pointed out in Hammond [1980].

For these reasons, it may not be too surprising that the use of unweighted total consumer surplus remains popular, for want of a good alternative. Also, Harberger [1971] has made an impassioned plea for using consumer surplus, in effect. One form of calculation he proposes also has the advantage that it relies only on data concerning the pre-project and post-project prices and quantities, without needing to know the precise shape of the various demand curves nor any demand elasticities (though it is a little unclear how post-project prices and quantities can be predicted in advance without some such knowledge). The use of consumer surplus as a good approximation has also recently been advocated by Willig [1976], whose work is called into question somewhat by Hausman [1982] and by Markandya [1978].

Consumer surplus, of course, is an exact measure of welfare for a single individual if all the goods for which prices change happen to have a zero income elasticity of demand. In this paper, I shall show

how to correct consumer surplus where the income elasticity of demand is not zero. The formulae to be developed will depend crucially on knowing income elasticities. However, this is not really such a serious limitation: income elasticities of demand have been studied in great detail ever since the work of Engel over a hundred years ago. Even if only a rough estimate is available, it is surely better to use this rough estimate than to assume that the income elasticity is zero, as consumer surplus implicitly does. Otherwise the formulae will also depend only on data concerning pre-project prices, quantities and incomes, as does the Harberger triangle measure of consumer surplus.

After correcting individual welfare measures for income effects, the next step is to construct a social welfare measure out of the individual welfare measures. The standard Harberger [1971] approach here is to take the unweighted sum of individual consumers' surpluses: this has the advantage that it relies only on data concerning prices and aggregate quantities, without any regard for the distribution of the aggregate between different individuals. However, this convenience is bought at the price of ethical unacceptability (as Harberger [1978] himself seems to realize and admit later on). Calculating unweighted total consumer surplus is not really a way either of avoiding making the interpersonal comparisons of utility which so many economists have found so abhorrent. Rather, such a calculation makes very special interpersonal comparisons to the effect that everybody's marginal dollar is worth the same, whether that person is extremely wealthy or a very poor person in need of a dollar's worth of medicine to arrest the progress of a debilitating disease. The social welfare measures presented below have a general set of welfare weights which can accomodate a wide range of ethical attitudes toward inequality and poverty. And the measures are also somewhat more accurate than the simple weighted sum of each individual's own welfare gains. They do rely on considerable cross-section information about different individuals' consumption before and after the project and about different individuals' income elasticities of demand. But calculations based on little more than guesswork concerning the differences between individuals seem to me far superior to the precise but crude use of unweighted total consumer surplus. That merely answers difficult, distributional questions by assuming that they are of no importance.

2. Approximate Calculations for One Consumer and Many Private Goods

Marshallian consumer surplus is really just a partial equilibrium tool, designed only to be used when the prices of most goods remain unaffected by the project which is being evaluated. True measures of benefit, however, can be developed also in a general equilibrium setting, in which the whole price vector may be affected by the project. True measures are derived from the consumer's expenditure or cost (of living) function. Their initial use for welfare calculations is essentially due to Hicks [1941, 1956], with notable elaboration and clarification by Klein and Rubin [1948] and McKenzie [1957].

Given the price vector p and a utility level u, the value of the expenditure function $E(p,u)$ is defined as that level of consumer expenditure which just enables the consumer to achieve utility level u when the price vector is p. $E(p,u)$ is the value of px when the commodity vector x is chosen to minimize px subject to the constraint that $U(x) \geqslant u$ -- i.e., the consumer's utility must not fall below u. The minimizing value of x here is $x(p,u)$, the compensated demand of the consumer when the price vector is p and income is varied to hold the consumer's utility constant at u.

Suppose the consumer faces the price vector p^o and has the commodity vector x^o yielding utility $u^o = U(x^o)$. Suppose too that x^o is expenditure minimizing, so $x^o = x(p^o,u^o)$ and $p^o \cdot x^o = E(p^o,u^o)$. Now, for any other price vector p, the minimum expenditure needed to achieve utility level u^o, namely $E(p,u^o)$, cannot be more than $p \cdot x^o$ because, if $p \cdot x^o$ is the consumer's expenditure, he can afford x^o which is a way of achieving utility u^o.

It follows that, for all p:

$$E(p,u^o) \leqslant p \cdot x^o \tag{2.1}$$

and so, for all p:

$$E(p,u^o) - p \cdot x^o \leqslant E(p^o,u^o) - p^o \cdot x^o = 0 \quad . \tag{2.2}$$

Thus, the function $E(p,u^o) - p \cdot x^o$ of the price vector p achieves a global maximum at p^o. Assuming $E(p,u^o)$ is differentiable at p^o, the first order condition for a maximum guarantees that:

$$\text{grad}_p\, E(p^o,u^o) = x^o \tag{2.3}$$

which is the well-known "envelope property" of the expenditure function.

Turning back to project evaluation, suppose that the consumer faces the price vector p^o without the project, buys the commodity bundle q^o, has income m^o and enjoys utility $u^o = U(q^o)$. With the project, suppose that the price vector becomes p^1, the commodity bundle q^1, income m^1 and utility $u^1 = U(q^1)$. In this case, the consumer's willingness to pay for the project is measured by:

$$B = m^1 - E(p^1,u^o) \tag{2.4}$$

because, if he faces prices p^1 and enjoys an income level of $m^1 - B = E(p^1,u^o)$ with the project, he is just able to afford to maintain his utility level at u^o, his utility without the project.

I shall now show how to calculate an approximation to (2.4) based on the data available - namely, p^o,q^o,m^o and p^1,q^1,m^1 - and on the vector estimate b^o of the income responsiveness of the consumer's demand for each good at p^o,q^o,m^o. Suppose, in fact, that the consumer's vector (uncompensated) demand function takes the linear form:

$$h(p,m) \equiv q^o + A^o(p - p^o) + b^o(m - m^o) \tag{2.5}$$

near (p^o,m^o). Then the Slutsky matrix of compensated demand responses or pure substitution effects will be:

$$S^o \equiv A^o + b^o(q^o)^T \tag{2.6}$$

near (p^o,u^o). The corresponding linearized compensated vector demand function is then (as an approximation valid near p^o):

$$x(p,u^o) \equiv q^o + S^o(p - p^o) \tag{2.7}$$

when utility is held fixed at u^o. Integrating the envelope formula (2.3), it then follows that the consumer's expenditure function takes the quadratic form:

$$E(p,u^o) = m^o + (p - p^o) \cdot q^o + \frac{1}{2} (p - p^o)^T S^o(p - p^o) \tag{2.8}$$

near p^o when utility is held fixed at u^o. Substituting (2.8) into (2.4) and using the formula (2.6) for the Slutsky matrix gives:

$$B = m^1 - m^o - (p^1 - p^o) \cdot q^o - \frac{1}{2} (p^1 - p^o)^T[A^o + b^o(q^o)^T](p^1 - p^o) \quad . \tag{2.9}$$

But, from (2.5) and the fact that $q^1 = h(p^1,m^1)$ it follows that:

$$q^1 = q^o + A^o(p^1 - p^o) + b^o(m^1 - m^o) \quad . \tag{2.10}$$

Substituting (2.10) into (2.9) to eliminate $A^o(p^1 - p^o)$:

$$\begin{aligned} B &= m^1 - m^o - \frac{1}{2} (p^1 - p^o) \cdot (q^o + q^1) + \frac{1}{2} (p^1 - p^o) \\ &\qquad \cdot b^o[m^1 - m^o - (p^1 - p^o) \cdot q^o] \\ &= \Delta m - \frac{1}{2} \Delta p \cdot (q^o + q^1) + \frac{1}{2} (\Delta p \cdot b^o) (\Delta m - \Delta p \cdot q^o) \quad . \end{aligned} \tag{2.11}$$

The Marshallian consumer surplus formula comes from assuming that $\Delta p \cdot b^o = 0$ - e.g. because all goods for which prices change have zero income elasticities. This gives:

$$B^M = \Delta m + CS \tag{2.12}$$

where

$$\text{(2.13)} \qquad CS \equiv -\frac{1}{2}\,\Delta p \cdot (q^o + q^1)$$

is exactly the Marshallian consumer surplus when the uncompensated demand curve happens to be linear.

The approximation in (2.11) can also be justified by replacing q^1 in (2.13) by $\hat{q}$, where the change from q^o to $\hat{q}$ represents the movement along the (linear approximation to) the <u>compensated</u> demand function rather than the uncompensated one. Thus

$$\text{(2.14)} \qquad B = \Delta m - \frac{1}{2}\,\Delta p \cdot (q^o + \hat{q}) \quad .$$

There are two possible procedures for computing an approximation to $\hat{q}$; fortunately, these two give the same answer. One procedure is to use (2.7) to derive the following approximation to the substitution effect:

$$\text{(2.15)} \qquad \hat{q} - q^o = S^o\,\Delta p \quad .$$

Alternatively, the consumer's change in real income is approximately $\Delta m - \Delta p \cdot q^o$ and this gives rise to an approximate income effect on demand of:

$$\text{(2.16)} \qquad q^1 - \hat{q} = b^o(\Delta m - \Delta p \cdot q^o) \quad .$$

These are consistent because, if (2.15) is added to (2.16), the result is:

$$\begin{aligned} \Delta q &= [S^o - b^o(q^o)^T]\,\Delta p + b^o\Delta m \\ &= A^o\Delta p + b^o\Delta m \end{aligned}$$

by (2.6), and this accords with (2.5). So, substituting from (2.16) in (2.14) (which is equivalent to but much more direct than substituting from (2.15) and then using (2.5) and (2.6) as well) gives:

$$B = \Delta m - \frac{1}{2}\Delta p \cdot [q^o + q^1 - b^o(\Delta m - \Delta p \cdot q^o)]$$

$$= \Delta m - \frac{1}{2}\Delta p \cdot (q^o + q^1) + \frac{1}{2}(\Delta p \cdot b^o)(\Delta m - \Delta p \cdot q^o)$$

as in (2.11). Alternatively, (2.11) can be written as:

$$B = \Delta m + CS + \frac{1}{2}(\Delta p \cdot b^o)(\Delta m - \Delta p \cdot q^o)$$

$$= B^M + \frac{1}{2}[\Delta p \cdot b^o(\Delta m - \Delta p \cdot q^o)]$$

$$= B^M + \frac{1}{2}\Delta p \cdot (q^1 - \hat{q})$$

using (2.13), (2.12) and (2.16). This makes it clear that (2.11) is a correction of (2.12) to allow for the income effect $q^1 - \hat{q}$.

All the above formulae represent partial equilibrium calculations which are appropriate for multiple price changes. For a true general equilibrium calculation, however, all quantity changes must be computed even if some prices do not change. Notice first that the identity $p \cdot h(p \cdot m) \equiv m$ implies from (2.5) that:

(2.17) $$p^o \cdot b^o = 1$$

then, because $m^o = p^o q^o$ and $m^1 = p^1 q^1$, (2.11) become:

$$B = \frac{1}{2}(p^1q^1 - p^oq^o - p^1q^o + p^oq^1) + \frac{1}{2}[(p^1 - q^o)^T b^o][p^1(q^1 - q^o)]$$

(2.18) $$= \frac{1}{2}(p^o + p^1) \cdot \Delta q + \frac{1}{2}(\Delta p \cdot b^o)(p^1 \cdot \Delta q)$$

$$= \frac{1}{2}[p^o + (p^1 \cdot b^o)p^1] \cdot \Delta q \quad \text{(using (2.17))} \quad .$$

These calculations are based on linear approximations to the compensated and uncompensated vector demand functions. They rely only on knowing the n-vector b^o of demand responses to income changes and not on the nxn matrix A^o of demand responses to price changes - though the prices and quantities induced by the project are difficult to predict without knowing A^o, of course.

3. One Consumer with Many Private and Public Goods

Many large public projects naturally involve providing public goods and it is therefore important to extend the scope of our welfare measures to accommodate such goods. To do so, let z denote the typical vector of publicly provided goods. The components of z may be quantity or quality measures, depending upon the context. Let the one consumer we continue to consider for the moment have utility function $U(x,z)$ in private goods x and public goods z. Given the price vector p, public goods vector z and utility level u, define the consumer's conditional expenditure function $E(p,z,u)$ as the level of expenditure on private goods which just enables the consumer to achieve utility level u at prices p when z is the public good vector. Then $E(p,z,u) \equiv p \cdot x(p,z,u)$ where $x(p,z,u)$ is the conditional compensated demand function of the consumer for private goods.

Arguing as in Section 2, it is easy to show that:

$$\text{grad}_p \, E(p^o,z^o,u^o) = x(p^o,z^o,u^o) \quad . \tag{3.1}$$

In addition:

$$(3.2) \qquad \text{grad}_z\, E(p^o,z^o,u^o) = -w^c(p^o,z^o,u^o)$$

where w is a vector with each component equal to the consumer's (compensated) marginal willingness to pay for the appropriate public good. The minus sign occurs because, if more desirable public goods are provided, the consumer requires less expenditure on private goods to achieve the same utility level.

Consider a project which alters the price vector p, public good vector z, utility level $u = U(x,z)$ and income m of the consumer respectively from p^o,z^o,u^o,m^o without the project to p^1,z^1,u^1,m^1 with the project. The consumer's willingness to pay for such a project is then given by:

$$(3.3) \qquad B = m^1 - E(p^1,z^1,u^o) \quad .$$

To calculate an approximation to B, postulate the following linear uncompensated conditional demand function for private goods:

$$(3.4) \qquad h(p,z,m) \equiv q^o + A^o_{pp}(p - p^o) + A^o_{pz}(z - z^o) + b^o(m - m^o)$$

and the following linear marginal willingness to pay function for public goods:

$$(3.5) \qquad w(p,z,m) \equiv w^o + A^o_{zp}(p - p^o) + A^o_{zz}(z - z^o) + c^o(m - m^o) \quad .$$

The corresponding mixed Slutsky-Antonelli matrix evaluated at (p^o,z^o,u^o) then has the following partitioned form:

$$\begin{pmatrix} S^o_{pp} & S^o_{pz} \\ S^o_{zp} & S^o_{zz} \end{pmatrix} = \begin{pmatrix} A^o_{pp} + b^o(q^o)^T & A^o_{pz} - b^o(w^o)^T \\ A^o_{zp} + c^o(q^o)^T & A^o_{zz} - c^o(w^o)^T \end{pmatrix} \tag{3.6}$$

or

$$S^o = A^o + \begin{pmatrix} b^o \\ c^o \end{pmatrix} \begin{pmatrix} q^o \\ -w^o \end{pmatrix}^T . \tag{3.7}$$

So linear approximations to the compensated demand function for private goods and marginal willingness to pay function for public goods, valid near (p^o,z^o,u^o), are:

$$\begin{aligned} x(p,z,u^o) &\equiv q^o + S^o_{pp}(p - p^o) + S^o_{pz}(z - z^o) \\ w^c(p,z,u^o) &\equiv w^o + S^o_{zp}(p - p^o) + S^o_{zz}(z - z^o) \quad . \end{aligned} \tag{3.8}$$

It follows that a quadratic approximation to $E(p,z,u\ )^o$ is:

$$\begin{aligned} E(p,z,u^o) &= m^o + (p - p^o)^T q^o - (z - z^o)^T w^o \\ &+ \frac{1}{2} \begin{pmatrix} p - p^o \\ -z + z^o \end{pmatrix}^T S^o \begin{pmatrix} p - p^o \\ z - z^o \end{pmatrix} \end{aligned} \tag{3.9}$$

Then

$$B = m^1 - E(p^1, z^1, u^o) \qquad \text{(from (3.3))}$$

$$= \Delta m - \Delta p \cdot q^o + \Delta z \cdot w^o - \frac{1}{2} \begin{pmatrix} \Delta p \\ -\Delta z \end{pmatrix}^T S^o \begin{pmatrix} \Delta p \\ \Delta z \end{pmatrix} \qquad \text{(from (3.9))}$$

$$= \Delta m - \Delta p \cdot q^o + \Delta z \cdot w^o - \frac{1}{2} \begin{pmatrix} \Delta p \\ -\Delta z \end{pmatrix}^T A^o \begin{pmatrix} \Delta p \\ \Delta z \end{pmatrix}$$

$$- \frac{1}{2} \begin{pmatrix} \Delta p \\ -\Delta z \end{pmatrix}^T \begin{pmatrix} b^o \\ c^o \end{pmatrix} \begin{pmatrix} q^o \\ -w^o \end{pmatrix}^T \begin{pmatrix} \Delta p \\ \Delta z \end{pmatrix} \qquad \text{(from (3.7))}$$

$$= \Delta m - \Delta p \cdot q^o + \Delta z \cdot w^o - \frac{1}{2} \begin{pmatrix} \Delta p \\ -\Delta z \end{pmatrix}^T \begin{pmatrix} \Delta q \\ \Delta w \end{pmatrix}$$

(3.10)

$$+ \frac{1}{2} \begin{pmatrix} \Delta p \\ -\Delta z \end{pmatrix}^T \begin{pmatrix} b^o \\ c^o \end{pmatrix} \left[\Delta m - \begin{pmatrix} q^o \\ -w^o \end{pmatrix}^T \begin{pmatrix} \Delta p \\ \Delta z \end{pmatrix}\right] \qquad \text{(from (3.4) and (3.5))}$$

$$= \Delta m - \Delta p \cdot q^o + \Delta z \cdot w^o - \frac{1}{2} \Delta p \cdot \Delta q + \frac{1}{2} \Delta z \cdot \Delta w$$

$$+ \frac{1}{2} (\Delta p \cdot b^o - \Delta z \cdot c^o) (\Delta m - \Delta p \cdot q^o + z \cdot w^o)$$

$$= \Delta m - \frac{1}{2} \Delta p \cdot (q^o + q^1) + \frac{1}{2} (w^o + w^1) \cdot \Delta z$$

$$+ \frac{1}{2} (\Delta p \cdot b^o - c^o \cdot \Delta z)(\Delta m - \Delta p \cdot q^o + w^o \cdot \Delta z) \quad .$$

The first three terms of (3.10) leave out the income effect and correspond to an approximate Marshallian consumer surplus calculation. The change in real income is approximately $\Delta m - \Delta p \cdot q^o + w^o \cdot \Delta z$ which leads to income effects on the demand for private goods and on the marginal willingness to pay for public goods given by:

$$\begin{pmatrix} q^1 - \hat{q} \\ w^1 - \hat{w} \end{pmatrix} = \begin{pmatrix} b^o \\ c^o \end{pmatrix} (\Delta m - \Delta p \cdot q^o + w^o \cdot \Delta z) \tag{3.11}$$

Then

$$B = \Delta m - \frac{1}{2} \Delta p \cdot (q^o + \hat{q}) + \frac{1}{2} (w^o + \hat{w}) \cdot \Delta z \tag{3.12}$$

where the move from (q^o, w^o) to $(\hat{q}, \hat{w})$ represents the substitution effect, to a linear approximation.

Once again, the calculation of (3.10) requires information only about the income responses (b^o, c^o); the details of the price responses and quantity responses in the matrix A^o are not required to be known, except insofar as they help to predict the quantities and prices if the project is introduced.

For a general equilibrium calculation, when the quantities of all private goods are included, the budget constraints $m^o = p^o \cdot q^o$, $m^1 = p^1 \cdot q^1$ and equation (2.17) -- $p^o \cdot b^o = 1$ -- can be imposed upon (3.10) to give:

$$\begin{aligned} B &= \frac{1}{2} (p^o + p^1) \cdot \Delta q + \frac{1}{2} (w^o + w^1) \cdot \Delta z \\ &\quad + \frac{1}{2} (p^1 \cdot b^o - 1 - c^o \cdot \Delta z) (p^1 \cdot \Delta q + w^o \cdot \Delta z) \\ &= \frac{1}{2} [p^o \cdot \Delta q + w^1 \cdot \Delta z + (p^1 \cdot b^o - c^o \cdot \Delta z)(p^1 \cdot \Delta q + w^o \cdot \Delta z)] \end{aligned} \tag{3.13}$$

which is a little simpler if less intuitively appealing than (3.10).

4. Many Individuals and Many Private Goods

To calculate social welfare benefits in a community of many individuals, it is obviously necessary to specify a social welfare function and to weight the gains which different individuals experience. This requires interpersonal comparisons of utility, to which many economists have been so averse. Yet the all too common procedure of calculating total unweighted consumer surplus also makes interpersonal comparisons of utility implicitly, and these particular comparisons have very little ethical appeal, as pointed out in the introduction.

Thus, I shall postulate a (Bergson [1938]) social welfare function of the form:

$$W \equiv F(U_1, U_2, \ldots, U_k) \tag{4.1}$$

where $U_1, U_2, \ldots, U_k$ is the vector of utilities of the k individuals in the community, and F is a differentiable function with positive partial derivatives everywhere. Write $\underset{\sim}{U}$ for this vector of utilities. Each individual i has an indirect utility function $v_i(p, m_i)$ which specifies i's utility when he chooses an optimal consumption bundle when faced with the price vector p and the income level m_i. It follows that social welfare can be expressed as a function of the price vector p and of the income profile $\underset{\sim}{m} = (m_1, m_2, \ldots, m_k)$:

$$\begin{aligned} W &\equiv F\big(v_1(p, m_1), v_2(p, m_2), \ldots, v_k(p, m_k)\big) \\ &\equiv V(p, \underset{\sim}{m}) \end{aligned}$$

where V is the <u>indirect</u> social function corresponding to F, given the utility functions of the k individuals of the community.

Consider now a project which takes the community from a price vector p^o, a distribution of consumption bundles $\underset{\sim}{q}^o = (q_1^o, q_2^o, \ldots, q_k^o)$ and an income distribution $\underset{\sim}{m}^o = (m_1^o, m_2^o, \ldots, m_k^o)$ to a new price vector p^1, consumption distribution $\underset{\sim}{q}^1 = (q_1^1, q_2^1, \ldots, q_k^1)$ and income distribution $\underset{\sim}{m}^1 = (m_1^1, m_2^1, \ldots, m_k^1)$. Let b_i^o denote individual i's vector of demand responses to income changes evaluated at (p^o, m_i^o). Then, as in equation (2.18) above, a measure of i's welfare gain is:

$$(4.3) \qquad B_i = \frac{1}{2} [p^o + (p^1 \cdot b_i^o) p^1] \cdot \Delta q_i \quad .$$

A first approximation to a measure of the social welfare benefit of the project can then be found by using welfare weights $\underset{\sim}{\beta}^o = (\beta_1, \beta_2, \ldots, \beta_k)$ determined as partial derivatives of the indirect social welfare function (4.2) so that for all i:

$$(4.4) \qquad \beta_i^o \equiv \frac{\partial V}{\partial m_i} (p^o, \underset{\sim}{m}^o)$$

and

$$(4.5) \qquad B = \Sigma_{i=1}^k \beta_i^o B_i$$

$$= \frac{1}{2} p^o \cdot \Sigma_{i=1}^k \beta_i^o \Delta q_i$$

$$+ \frac{1}{2} \Sigma_{i=1}^k \beta_i^o (p^1 \cdot b_i^o)(p^1 \cdot \Delta q_i)$$

Such a calculation depends on cross-section information concerning the quantity changes Δq_i and the income-responsiveness of demands $b_{i,}^o$ for each individual, as well as on the welfare weights β_i^o. It therefore requires considerably more information than the corresponding calculation based on unweighted total Marshallian consumer surplus, which is (see (2.12) and (2.13)):

$$B^M = \Sigma_i \Delta m_i - \frac{1}{2} \Delta p \cdot \Sigma_i (q_i^o + q_i^1) \tag{4.6}$$

and which therefore depends only on aggregate income and demands with and without the project, and not on the distribution of that income or those demands. However, in a community where there is serious inequality or poverty, the details involved in the more complicated calculation of (4.5) can be of vital importance. Total consumption $\Sigma_i \Delta q_i$ could rise and yet particularly poor individuals simultaneously might be driven below the margin of survival. Even in less dramatic instances, (4.6) depends on very particular interpersonal comparisons whereas (4.5) is much more flexible, as well as correcting properly for income effects.

Formula (4.5), however, is only a first approximation which can be improved by taking account of second order effects on the indirect social welfare function. To do this, I shall follow Sen [1976, 1979] and treat the community as one large consumer interested in each individual's consumption of each good. Thus, given the profile $\underset{\sim}{x} = (x_1, x_2, \ldots, x_k)$ of the k individuals' commodity vectors, social welfare--the "utility" of the large consumer--is, using (4.1):

$$W(\underset{\sim}{x}) \equiv F(U_1(x_1), U_2(x_2), \ldots, U_k(x_k)) \tag{4.7}$$

I shall assume that W here is quasi-concave as a function of the consumption profile $\underset{\sim}{x}$ - as pointed out by Negishi [1963], this will be true provided that F is quasi-concave and each individual's utility function U_i is concave (since F is increasing, as assumed earlier). Then the consumption distribution $\underset{\sim}{q}$ associated with prices p and an income distribution $\underset{\sim}{m}$ (with $m_i = p \cdot q_i$ for all i) maximizes W subject to a budget constraint of the form:

$$(4.8) \qquad \Sigma_{i=1}^{k} \beta_i p \cdot x_i < \Sigma_{i=1}^{k} \beta_i m_i$$

where $\beta_i = (\partial V/\partial m_i)(p,\underset{\sim}{m})$ is the welfare weight attached to i's income. Thus the community behaves as a single consumer achieving a consumption distribution $\underset{\sim}{q}$ at prices $\underset{\sim}{\beta} p^T$ and income $m = \Sigma_{i=1}^{k} \beta_i m_i = \Sigma_{i=1}^{k} \beta_i p q_i$.1/ To estimate the welfare benefit of a project to the community, the obvious approach is to adapt the formula (2.18) for a single consumer to the community as a whole, treating different individuals' consumption as different goods in the community "utility" function which represents the measure of social welfare. Thus q^o, q^1 are replaced by the distributions $\underset{\sim}{q}^o, \underset{\sim}{q}^1$ respectively, and the price vectors p^o, p^1 by $\underset{\sim}{\beta}^o p^{oT}$, $\underset{\sim}{\beta}^1 p^{1T}$ respectively. It then remains to calculate the appropriate replacement for b^o the vector of demand responses to income changes. For good g and consumer i, the appropriate response of x_{ig}, i's consumption of good g, to the change in aggregate income $m = \Sigma_i \beta_i^o m_i$, with relative welfare weights $\underset{\sim}{\beta}^o$ and relative commodity price p^o held constant, is given by:

$$(4.9) \qquad \underset{\sim}{b}_{ig} = b_{ig} \frac{\partial m_i}{\partial m}$$

where $b_{ig} = \partial x_{ig}/\partial m_i$, the ordinary response of i's demand for good g to a change in i's own income. Thus the vector $\underset{\sim}{\mu}$ of responses, $\mu_i \equiv \partial m_i/\partial m$, of i's income m_i to a change in welfare weighted total income m must be calculated. This can be done by noticing that, because the relative welfare weights $\underset{\sim}{\beta}^o$ are held constant, by assumption, there must be some positive number λ independent of i for which, all i:

$$\frac{\partial V}{\partial m_i}(p^o, \underset{\sim}{m}) = \lambda \beta_i^o \tag{4.10}$$

while, in addition, of course:

$$m = \Sigma_i \beta_i^o m_i \ . \tag{4.11}$$

Differentiating these equations totally gives:

$$\begin{aligned} &\Sigma_j \frac{\partial^2 V}{\partial m_i \partial m_j}(p^o, \underset{\sim}{m}^o) dm_j = d\lambda \beta_i^o \quad (i = 1 \text{ to } k) \ ; \\ &dm = \Sigma_i \beta_i^o dm_i \ . \end{aligned} \tag{4.12}$$

Writing H for the Hessian matrix $[\partial^2 V/\partial m_i \partial m_j]$ of partial second derivatives of the indirect social welfare function V, and assuming that H has an inverse, it follows that $\underset{\sim}{dm} = H^{-1}\underset{\sim}{\beta}^o d\lambda$ and that $dm = \underset{\sim}{\beta}^{oT} H^{-1} \underset{\sim}{\beta}^o d\lambda$ so $d\lambda = dm/\underset{\sim}{\beta}^{oT} H^{-1} \underset{\sim}{\beta}$ and, finally:

$$\underset{\sim}{\mu}^o = \frac{H^{-1}\underset{\sim}{\beta}^o}{\underset{\sim}{\beta}^{oT} H^{-1} \underset{\sim}{\beta}^o} \ . \tag{4.13}$$

A very important special case is when the direct social welfare function and the indirect social welfare function both have corresponding additively separable forms:

$$W(\underset{\sim}{x}) \equiv \Sigma_{i=1}^{k} U_i(x_i), \quad V(p,\underset{\sim}{m}) \equiv \Sigma_{i=1}^{k} v_i(p,m_i) \tag{4.14}$$

of the kind strongly advocated recently by Mirrlees [1982], for instance. Then the Hessian matrix H is diagonal, with its diagonal terms given by the total second derivatives v_i'' of the individuals' indirect utility functions which appear in (4.14). Thus, corresponding to (4.13):

$$\mu_i^o = \frac{\beta_i^o (v_i'')^{-1}}{\Sigma_{j=1}^{k} (\beta_j^o)^2 (v_j'')^{-1}} \tag{4.15}$$

in this case. But $\beta_i = v_i'$ and so, writing $\eta_i \equiv -v_i'/v_i''$ for each individual i, and η_i^o for the ratio of these derivatives evaluated at (p^o,m_i^o), equation (4.8) can be written as:

$$\mu_i^o = \frac{\eta_i^o}{\Sigma_{j=1}^{k} \beta_j^o \eta_j^o} \tag{4.16}$$

By analogy with the Arrow-Pratt theory of risk-bearing, Atkinson [1970] suggests interpreting the elasticity of marginal utility $-\ m_i v_i''/v_i'$ as the coefficient of relative inequality aversion, and Kolm [1976] suggests that $-v_i''/v_i'$ be interpreted as the coefficient of absolute inequality aversion. This prompts the interpretation of η_i, the reciprocal of the coefficient of absolute inequality aversion, as the coefficient of "inequality tolerance" (by analogy with risk tolerance). In any case, however η_i is interpreted, the calculation of η_i^o from any assumed additively separable form of indirect social welfare function (4.14) is just routine.

After this necessary diversion to show the partial derivatives $\partial m_i/\partial m$ appearing in (4.9) can be evaluated, it is now possible to give formulae for a better approximate measure of social welfare benefits than the first approximation (4.5) afforded. Making the appropriate replacements in the formula (2.18) for individual benefit leads to:

$$(4.17) \qquad B = \frac{1}{2}\, p^o \cdot \Sigma_i \beta_i^o \Delta q_i + \frac{1}{2}\,(p^1 \cdot \Sigma_j \beta_j^1 b_j^o \mu_j^o) p^1 \cdot \Sigma_i \beta_i^1 \Delta q_i\,,$$

where each μ_j^o is to be calculated from (4.13) or, in the special additively separable case, from (4.16).

Compared with the earlier first approximation in (4.5), the extra information required by this second approximation consists of the welfare weights $\underset{\sim}{\beta}^1$ for the income distribution after the project (as well as $\underset{\sim}{\beta}^o$ which was required earlier), and the "inequality tolerances" μ_i^o $(i = 1$ to $k)$ associated with the indirect social welfare function (4.2). Once again, making even crude guesses of the appropriate values of these parameters seems far superior to ignoring them altogether.

Notice now that the calculated "willingness to pay" for a change B has no significance other than its sign. Because of changing welfare weights, a general equilibrilum calculation is almost inevitable, and this requires specifying how the costs as well as the benefits of the project will be distributed between different individuals. When all this has been done, the sign of B, a measure of net benefit, determines whether or not the project is beneficial. Since the measure of B is only approximate, caution is necessary if B is small in absolute value.

5. Many Individuals with Private and Public Goods

For relative simplicity of notation, the analysis in Section 4 was carried for an economy without public goods. Using the results of Section 3, however - particularly the approximate benefit measure (3.13) --it is easy to allow for public goods in an approximate measure of social welfare benefits such as (4.5) or the more accurate (4.17). Each individual has an indirect utility function $v_i(p,z,m_i)$ which depends now on the public good vector z as well. Given a welfare function such as (4.1), the indirect social welfare function takes the form:

$$W \equiv F\big(v_1(p,z,m_1),v_2(p,z,m_2),\ldots,\ v_k(p,z,m_k)\big) = V(p,z,\underset{\sim}{m}) \tag{5.1}$$

which also depends on public goods. The welfare weights $\underset{\sim}{\beta}^o$ are the partial derivatives $\partial V/\partial m_i$ of this function evaluated at $(p^o,z^o,\underset{\sim}{m}^o)$ and similarly for $\underset{\sim}{\beta}^1$. The formula corresponding to (4.5), derived from (3.13), is:

$$\begin{aligned} B &= \frac{1}{2}\,[p^o \cdot \Sigma_i \beta_i^o \Delta q_i + \Sigma_i \beta_i^o w_i^1 \cdot \Delta z] \\ &\quad + \frac{1}{2}\,\Sigma_i \beta_i^o (p^1 \cdot b_i^o - c_i^o \cdot \Delta z)(p^1 \cdot \Delta q_i + w_i^o \cdot \Delta z) \quad . \end{aligned} \tag{5.2}$$

If the income effects are ignored and the welfare weights taken all equal, then the corresponding approximate measure of total unweighted Marshallian consumer surplus is (from (3.10)):

$$(5.3) \qquad B^M = \Sigma_i \Delta m_i - \frac{1}{2} \Delta p \cdot \Sigma_i (q_i^o + q_i^1) + \frac{1}{2} \Sigma_i (w_i^o + w_i^1) \cdot \Delta z \quad .$$

Like (4.5), this depends only on aggregates in a convenient manner. Indeed, $\Sigma_i w_i^o$ and $\Sigma_i w_i^1$ represent precisely the Lindahl-Samuelson measures of marginal benefits for public goods. But like (4.6), this measure has no ethical justification at all.

To derive a more accurate approximation than (5.2), consider the society as an "aggregate consumer" with utility function:

$$(5.4) \qquad W(\underset{\sim}{x},z) \equiv F(U_1(x_1,z),U_2(x_2,z),\ldots,U_k(x_k,z))$$

Assuming as before that W is concave in $\underset{\sim}{x}$, a distribution $\underset{\sim}{q}$ of private goods associated with prices p, income distribution $\underset{\sim}{m}$ and public good vector z will maximize $W(\underset{\sim}{x},z)$ with respect to $\underset{\sim}{x}$ subject to the budget constraint (4.8):

$$(5.5) \qquad \Sigma_i \beta_i p x_i \leq \Sigma_i \beta_i m_i$$

where $\beta_i \equiv \partial V(p,z,\underset{\sim}{m})/\partial m_i$. This "aggregate consumer's" marginal willingness to pay for public goods h is then given by:

$$(5.6) \qquad \begin{aligned} w_h &= \frac{\partial w}{\partial z_h} = \Sigma_i \frac{\partial F}{\partial U_i} \cdot \frac{\partial V_i}{\partial z_h} \\ &= \Sigma_i \frac{\partial V}{\partial m_i} \cdot \frac{\partial v_i/\partial z_h}{\partial v_i/\partial m_i} = \Sigma_i \beta_i w_{ih} \end{aligned}$$

where w_{ih} is consumer i's marginal willingness to pay for public good h. So, as a vector:

$$w = \Sigma_i \beta_i w_i \quad . \tag{5.7}$$

We also need to calculate $c_h^o \equiv \partial w_h/\partial m$ for each public good h, where $m \equiv \Sigma_i \beta_i m_i$. In fact:

$$c_h^o = \frac{\partial w_h}{\partial m} = \Sigma_i \beta_i^o \frac{\partial w_{ih}}{\partial m_i} \frac{\partial m_i}{\partial m} = \Sigma_i \beta_i^o \cdot c_{ih}^o \mu_i^o \tag{5.8}$$

where

$$c_{ih}^o \equiv \frac{\partial w_{ih}}{\partial m_i} \quad (\text{all} \quad i,h) \tag{5.9}$$

and $\mu_i^o = \partial m_i/\partial m$ is calculated using formulae akin to (4.13) or, in the special additive case, (4.16). Substituting all this into an appropriate form of (3.13) for the whole society leads finally to:

$$\begin{aligned} B = {} & \frac{1}{2}\,(p^o \cdot \Sigma_i \beta_i^o \Delta q_i + \Sigma_i \beta_i^1 w_i^1 \cdot \Delta z) \\ & + \frac{1}{2}\,[\Sigma_j \beta_j^o \mu_j^o (p^1 \cdot b_j^o - c_j^o \cdot \Delta z)](p^1 \cdot \Sigma_i \beta_i^1 \Delta q_i + \Sigma_i \beta_i^o w_i^o \cdot \Delta z) \quad . \end{aligned} \tag{5.10}$$

6. Bibliographical Notes and Conclusion

I have presented approximate measures of social welfare benefits from a large project which affects relative prices and public good quantities significantly. They are essentially quadratic approximations derived from integrating linear compensated demand functions. These compensated demand functions are in turn estimated by combining assumed knowledge of responses to income changes with linear interpolation between the price and quantity pairs with and without the project (also assumed known, as is common in the literature on welfare measurement).

Similar quadratic approximations have been presented by Harberger [1964a, 1964b], Bergson [1973], Green and Sheshinski [1979] etc., often in connection with tax reforms rather than public projects. The same principles of measurement apply, however. Green and Sheshinski especially are concerned with measures of deadweight loss, a topic which is well treated in Kay [1980] and Zabalza's [1982] improvements of the earlier article by Diamond and McFadden [1974]. Most of this work, however, treats only one consumer and ignores public goods. It also assumes that the Slutsky matrix of the consumer is known at the allocation that results without the project (or tax reform). Section 2 shows how it suffices to know only the responses of demand to income changes as well as the details of the allocation that results from the project (or tax reform).

The methods of Seade [1978] also rely on Engel curves being (approximately) linear but his welfare measures depend on complete knowledge of price responses and so require even more information than the approximations discussed in the previous paragraph. Approximations of an arbitrarily high order have been considered by McKenzie and Pearce [1976, 1982]. This assumes that the consumer's expenditure function to be continuously differentiable a suitably large number of times, and requires knowledge of high order derivatives as well.

Exact measures of benefit for one consumer have been discussed recently by Diamond and McFadden [1974], Hause [1975] and Hausman

[1981], amongst others, using the methods which go back to Hicks and which underlie equation (2.4) of this paper. Hausman [1981] in particular shows how to calculate benefit exactly if the consumer's demand function takes the linear form (2.5) exactly when just one price is variable. In fact, if (2.5) is exact, then the compensated demand function is necessarily nonlinear so that (2.7) is only an approximation. As I intend to show elsewhere, however, Hausman's exact calculations do not work well when more than one price is variable--or certainly more than two. If (2.5) holds exactly when several prices vary, Slutsky symmetry imposes unacceptably strong restrictions. The same is true for the loglinear form of (2.5) which Hausman also considers. Finding a flexible functional form which allows an exact calculation of benefit from knowledge of prices and quantities with and without the project and of income responses at one point is a complicated matter which I shall leave for a later paper. For practical applications, the kind of approximations presented here are probably adequate in any case.

For a community of many individuals, suitable exact formulae with <u>fixed</u> welfare weights were proposed by Bergson [1980]. However, corresponding approximations with fixed welfare weights can be improved upon, as in the more accurate approximation represented in equation (5.10), for which some information about second derivatives of the welfare function is needed. Of course, it is also necessary to have a great deal of cross-section information to compute social welfare benefits properly. My claim is that, even where this information is seriously lacking, as will no doubt often be the case, it is far better to make as intelligent guesses as possible concerning the required information than to ignore the problem altogether by using some crude measure such as unweighted total consumer surplus. And in fact the information needed for such a measure to be calculated properly is actually not so much less, since it is hard to know what prices and aggregate quantities will be with the project in the absence of any cross-section information because of the usual problems of aggregation. Thus the only really new requirements are the welfare judgements, and here it seems to me better to be explicit and to allow much more reasonable judgments than those which lie behind unweighted total consumer surplus. Even without such welfare judgments, moreover, one can use formula (3.10) or (3.13) to calculate individual welfare benefits for different classes of individuals, and so calculate the likely distribution of benefits. This is an important exercise in its own right.

The approximate measures developed here suffer from one major limitation. They have been developed for public goods and for private goods which consumers are free to buy (and sell) at fixed competitive prices. These prices can be gross of tax provided that all consumers face the same tax on any good subject to taxation. What is specifically excluded are goods (including labour supply) subject to rationing constraints which are not the same for everybody, or indeed any goods which are charged for according to nonlinear pricing schemes. The development of appropriate approximate benefit measures for the many important goods of this kind must be left for later work.

Footnotes

1/ Both $\underset{\sim}{q}$ and $\underset{\sim}{\beta p}^T$ are $k \times n$ matrices, in effect, where k is the number of individual consumers and n is the number of commodities. However, the total value of $\underset{\sim}{q}$ at prices $\underset{\sim}{\beta p}^T$ takes the form of a scalar product $\Sigma_{i=1}^{k} \beta^i p \cdot q^i$.

Notice that now only general equilibrium calculations are going to be of use; because welfare weights will change, so will all the relevant prices.

Also, it is <u>not</u> being assumed that any deliberate maximization of W takes place. Consumers buy and sell freely at market clearing prices p. The mechanism which actually determines income distribution only matters insofar as it affects m^o and m^1. The prices $\beta_i \cdot p_g$ are "virtual" or "shadow" prices rather like those which "explain" the behaviour of a rational consumer. In fact, $\beta_i - \beta_j$ is precisely the shadow price of whatever constraint has prevented the transfer of income form j to i (assuming $\beta_i > \beta_j$).

References

Atkinson, A.B. [1970], "On the Measurement of Inequality," Journal of Economic Theory, 2, pp. 244-263.

Bergson, A. [1938], "A Reformulation of Certain Aspects of Welfare Economics," Quarterly Journal of Economics, 52, pp. 310-334.

Bergson, A. [1973], "On Monopoly Welfare Losses," American Economic Review, 63, pp. 853-870.

Bergson, A. [1980], "Consumer's Surplus and Income Redistribution," Journal of Public Economics, 14, pp. 31-47.

Dasgupta, P. S., S. A. Marglin and A. K. Sen [1972], Guidelines for Project Evaluation. New York: U.N.I.D.O.

Diamond, P.A. and D. McFadden [1974], "Some Uses of the Expenditure Function in Public Finance," Journal of Public Economics, 3, pp. 3-21.

Dixit, A. K. and P. A. Weller [1979], "The Three Consumer's Surpluses," Economica, 46, pp. 125-135.

Green, J. R. and E. Sheshinski [1979], "Approximating the Efficiency Gain of Tax Reforms," Journal of Public Economics, 11, pp. 179-195.

Hammond, P. J. [1980], "Cost Benefit Analysis as a Planning Procedure," Ch. 8, pp. 221-249 of Contemporary Economic Analysis, Vol. 2. London: Croom-Helm.

Harberger, A. C. [1964a], "The Measurement of Waste," American Economic Review, Papers and Proceedings, 54, pp. 58-76.

Harberger, A. C. [1964b], "Taxation, Resource Allocation and Welfare," The Role of Direct and Indirect Taxes in the Federal Revenue System, N.B.E.R., pp. 25-70. Princeton: Princeton University Press.

Harberger, A. C. [1971], "Three Basic Postulates for Applied Welfare Analysis: An Interpretive Essay," Journal of Economic Literature, 9, pp. 785-797.

Harberger, A. C. [1978], "On the Use of Distributional Weights in Social Cost Benefit Analysis," Journal of Political Economy, 86, pp. S87-S120.

Harris, R. G. [1978], "On the Choice of Large Projets," Canadian Journal of Economics, 11, pp. 404-423.

Hause, J. C. [1975], "The Theory of Welfare Cost Measurement," Journal of Political Economy, 83, pp. 1145-1182.

Hausman, J. A. [1981], "Exact Consumer's Surplus and Deadweight Loss," American Economic Review, 71, pp. 662-676.

Hicks, J. R. [1941], "The Rehabilitation of Consumers' Surplus," Review of Economic Studies, 8, pp. 108-116.

Hicks, J. R. [1956], A Revision of Demand Theory. Oxford: Clarendon Press.

Kay, J. A. [1980], "The Deadweight Loss from a Tax System," Journal of Public Economics, 13, pp. 111-119.

Klein, L.R. and H. Rubin [1948], "A Constant-Utility Index of the Cost of Living," Review of Economic Studies, 15, pp. 84-87.

Kolm, S. C. [1976], "Unequal Inequalities, I and II," Journal of Economic Theory, 12, pp. 416-442, and 13, pp. 82-111.

Little, I. M. D. and J. A. Mirrlees [1968], Manual of Industrial Project Analysis in Developing Countries, Vol. II: Social Cost Benefit Analysis. Paris: O.E.C.D.

Little, I. M. D. and J. A. Mirrlees [1974], Project Appraisal and Planning for Developing Countries. London: Heinemann.

McKenzie, G. W. and I. F. Pearce [1976], "Exact Measures of Welfare and the Cost of Living," Review of Economic Studies, 43, pp. 465-468.

McKenzie, G. W. and I. F. Pearce [1982], "Welfare Measurement: A Synthesis," American Economic Review, 72, pp. 669-682.

McKenzie, L. W. [1957], "Demand Theory without a Utility Index," Review of Economic Studies, 24, pp. 185-189.

Markandya, A. [1978], "The Quality of Current Approximations to the Measurement of Compensation Costs," Oxford Economic Papers, 30, pp. 423-433.

Mirrlees, J. A. [1982], "The Economic Uses of Utilitarianism," A. K. Sen and B. Williams (eds.) Utilitarianism and Beyond, Chap. 3, pp. 63-84. Cambridge: Cambridge University Press.

Negishi, T. [1963], "On Social Welfare Function," Quarterly Journal of Economics, 77, pp. 156-158.

Negishi, T. [1972], General Equilibrium Theory and International Trade. Amsterdam: North-Holland.

Samuelson, P. A. [1947],] Foundations of Economic Analysis. Cambridge, Mass.: Harvard University Press.

Seade, J. K. [1978], "Consumer's Surplus and the Linearity of Engel Curves," Economic Journal, 88, pp. 511-523.

Sen, A. K. [1976], "Real National Income," Review of Economic Studies, 43, pp. 19-39.

Sen, A. K. [1979], "The Welfare Basis of Real Income Comparisons: A Survey," Journal of Economic Literature, 17, pp. 1-45.

Willig, R. [1976], "Consumer Surplus without Apology," American Economic Review, 66, pp. 589-597.

Zabalza, A. [1982], "Compensating and Equivalent Variations, and the Deadweight Loss of Taxation," Economica, 49, pp. 355-359.