The Steady-State Growth Theorem: Understanding Uzawa (1961)

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November 16, 2005

This note revisits the proof of the Steady-State Growth Theorem, first given by Uzawa in 1961. We provide a clear statement of the theorem and a new version of Uzawa's proof that makes the intuition underlying the result more apparent. For example, in the special case of factor-augmenting technical change, i.e. $Y_t = F(B_t K_t, A_t L_t)$, the effective inputs BK and AL must grow at the same rate in steady state; otherwise trends in the factor shares are induced. The fact that effective inputs must "balance" suggests a new interpretation of balanced growth. Because capital accumulates and therefore inherits the trend in AL, the balance condition implies that technical change must be purely labor augmenting.

Key Words: Steady state, balanced growth, labor-augmenting technical change.
JEL Classification: O40, E10

1. INTRODUCTION

The Steady-State Growth Theorem says that if a neoclassical growth model exhibits steady-state growth, then technical change must be labor augmenting, at least in steady state.[1] It did not escape the attention

*Jones thanks the Toulouse Network on Information Technology for research support.

[1] It is sometimes added that an alternative is for the production function to be Cobb-Douglas, at least in steady state. But this is really subsumed in the original version of

of economists, either in the 1960s or more recently, that this is a very restrictive theorem. We often want our models to exhibit steady-state growth, but why should technical change be purely labor augmenting? The induced-innvoation literature associated with Fellner (1961), Kennedy (1964), Samuelson (1965), and Drandakis and Phelps (1966) explicitly pondered this question without achieving a clear answer. Recently, Acemoglu (2003) and Jones (2005) have returned to this puzzle.

Perhaps surprisingly, then, given its importance in the growth literature, we have been unable to find a clear statement and proof of the theorem. In addition, exactly why the result holds is not something that is well understood. What is the intuition for why technical change must be labor augmenting?

Uzawa (1961) is typically credited with the proof of the result,[2] and there is no doubt that he proved the theorem. However, Uzawa is primarily concerned with showing the equivalence of *Harrod-neutral* technical change (i.e. technical change that leaves the capital share unchanged if the interest rate is constant) and labor-augmenting technical change, formalizing the graphical analysis of Robinson (1938). It is of course, only a small and well-known step to show that steady-state growth requires technical change to be Harrod neutral. But the modern reader of Uzawa will be struck by two things. First is the lack of a statement and direct proof of the steady-state growth theorem. Second is the absence of economic intuition, both in the method of proof and more generally in the paper.

Barro and Sala-i-Martin (1995, Chapter 2) come close to providing a clear statement and proof of the theorem. However, their statement of the theorem is more restrictive: if technical change is factor augmenting at a constant exponential rate, then steady-state growth requires it to be labor

the theorem since technical change can always be written in the labor-augmenting form in steady state if the production function is Cobb-Douglas.

[2]For example, see Barro and Sala-i-Martin (1995) and Solow (1999).

augmenting. This leaves the door open to the possibility that there might be some perverse non-factor augmenting twist of technical change that could be consistent with steady-state growth. McCallum (1996) also comes close, providing a proof of the general theorem very similar to Uzawa's approach; by sticking so closely to Uzawa, however, the intuition for the result remains elusive.[3]

This comment fills the gap in the literature. We provide a clear statement and proof of the steady-state growth theorem. The inspiration for the proof is Uzawa (1961), but we present the crucial steps in a slightly different way that allows the economic intuition for the proof to come through.

2. STATING THE THEOREM

The steady-state growth theorem applies to the one-sector neoclassical growth model. We begin by defining the model precisely and then defining a balanced growth path.

DEFINITION 2.1. A neoclassical growth model is given by the following economic environment:

$$Y_t = F(K_t, L_t; t), \tag{1}$$

$$\dot{K}_t = Y_t - C_t - \delta K_t, \;\; K_0 > 0, \;\; \delta \geq 0, \tag{2}$$

and

$$L_t = L_0 e^{nt}, \;\; L_0 > 0, \;\; n \geq 0. \tag{3}$$

The production function F satisfies the standard neoclassical properties: constant returns to scale in K and L, positive and diminishing marginal products of K and L, and the Inada conditions that the marginal product of a factor input goes to zero as that input goes to infinity and goes to infinity as the input goes to zero.

[3] Motivated by a previous version of our paper, Russell (2004) provides a quick mathematical proof of the theorem that exploits some methods from the physics literature on a class of partial differential equations called advective equations.

A balanced growth path in the neoclassical growth model is defined as a situation in which all quantities grow at constant exponential rates (possibly zero) forever. We follow the usual convention of also refering to this as a steady state.

Finally, we will define $F_K K/Y$ to be the capital share and $F_L L/Y$ to be the labor share. As usual, the two factor shares sum to a value of unity, by Euler's theorem. We follow standard notation in denoting $y \equiv Y/L$ and $k \equiv K/L$, and we will use an asterisk superscript to denote a variable along the steady-state path.

With these definitions, we can now present the Steady-State Growth Theorem:

THEOREM 2.1 (The Steady-State Growth Theorem, Uzawa 1961). *If a neoclassical growth model possesses a steady state with constant, nonzero factor shares and $\dot{y}_t^*/y_t^* = g > 0$, then it must be possible along the steady-state path to write the production function as $Y_t^* = G(K_t^*, A_t L_t)$, where $\dot{A}_t/A_t = g$ and where G is a neoclassical production function.*

As is well-known, in the case of Cobb-Douglas production, capital- and labor-augmenting technical change are equivalent. One sometimes sees the theorem interpreted as saying that technical change must be labor augmenting or the production function must be Cobb-Douglas. This is equivalent to the statement of the theorem as given.

3. PROVING THE THEOREM

This proof largely follows Uzawa (1961) in spirit. It differs in that we provide more economic intuition, highlight the key steps of the proof more clearly, and fill in some details. The main innovation in the proof is in writing the key differential equation in (4) below in terms of the elasticity of output with respect to the capital-output ratio. This produces a familiar equation in a way that Uzawa's consideration of the marginal product of capital does not.

The capital-output ratio is a key variable throughout the proof, so we define $x \equiv K/Y$. We also make the standard definition $f(k;t) \equiv F(k,1;t)$.

The proof now follows.

1. The first step of the proof is to rewrite the production function in terms of the capital-output ratio: $y_t = \phi(x_t;t)$. Intuitively, this step is readily understood by drawing the production function in (k,y) space: for each ray through the origin — that is for each capital-output ratio — there is a unique level of output per worker on that ray.[4]

2. Next, we note that the elasticity of y_t with respect to x_t satisfies a familiar property:

$$\frac{\partial \log y_t}{\partial \log x_t} = \frac{\alpha(x_t;t)}{1-\alpha(x_t;t)} \tag{4}$$

where $\alpha(x_t;t) \equiv f_k k_t/y_t$ is the capital share. This equation says that the elasticity of output with respect to the capital-output ratio is equal to the ratio of the capital and labor shares. Such an equation is well-known in the case of Cobb-Douglas production, where it has been exploited by Mankiw, Romer and Weil (1992), Klenow and Rodriguez-Clare (1997), and Hall and

[4] Formally, we can use the inverse function theorem to justify this step. The capital-output ratio depends only on k_t and t, since y_t is a function of k_t and t: $x_t = k_t/y_t = k/f(k_t;t) \equiv h(k_t;t)$. We can apply the inverse function theorem to show that this function can be inverted:

$$\begin{aligned}\frac{\partial h(k_t;t)}{\partial k_t} &= \frac{1}{f(k;t)} - \frac{k_t f_k(k_t;t)}{f(k_t;t)^2} \\ &= \frac{1}{f(k_t;t)}\left(1 - \frac{f_k(k_t;t)k_t}{f(k_t;t)}\right) \\ &\neq 0 \ \forall k_t,\end{aligned}$$

where the last step follows from the fact that the labor share is strictly between zero and one. Therefore, by the inverse function theorem, $h^{-1}(\cdot\,;t)$ exists, and we can write $k_t = h^{-1}(x_t;t)$. Finally, we can substitute this result into the production function to get $y_t = f(k_t;t) = f(h^{-1}(x_t;t),t) \equiv \phi(x_t;t)$.

Jones (1999), among others. Equation (4) shows that this property holds more generally.[5]

3. Now comes the key step of the proof. From this point on, we assume the economy is on a balanced growth path. Because the capital share is constant in steady state, the right side of equation (4) is invariant over time. Then, since $y_t = \phi(x_t; t)$, we can write this equation as

$$\frac{\partial \log \phi(x^*; t)}{\partial \log x^*} = \frac{\alpha(x^*)}{1 - \alpha(x^*)}, \tag{5}$$

where we use an asterisk to indicate a quantity along a balanced growth path.

Because the right-hand side of this equation does not depend on time, this partial differential equation can be solved to yield[6]

$$\log \phi(x^*; t) = a(t) + \int \frac{\alpha(x^*)}{1 - \alpha(x^*)} \frac{dx^*}{x^*} \tag{6}$$

for some function $a(t)$. And therefore

$$y_t^* = \phi(x^*; t) = A(t)\psi(x^*), \tag{7}$$

where $A(t) \equiv \exp(a(t)) > 0$ and $\psi(x^*) \equiv \exp\left(\int \frac{\alpha(x^*)}{1-\alpha(x^*)} \frac{dx^*}{x^*}\right)$.

This is the crucial result. We've shown that the effects of t and x^* can be separated. This implies, for example, the familiar result that $y_t^*/A_t = \psi(x^*)$ is constant along a balanced growth path, where $A_t \equiv A(t)$. Since y_t^* grows at rate g by assumption, it must therefore be the case that $\dot{A}_t/A_t = g$ as well.

[5]To derive this equation, notice that $y_t = f(k_t; t) = \phi(x_t; t) = \phi(h(k_t; t); t)$. Differentiating gives

$$\frac{\partial f(k_t; t)}{\partial k_t} = \frac{\partial \phi(x_t; t)}{\partial x_t} \cdot \frac{\partial h(k_t; t)}{\partial k_t}.$$

Using the expression for $\frac{\partial h}{\partial k}$ in footnote 4 above and rearranging gives the equation in the main text.

[6]This can be readily verified by differentiating the solution.

4. To conclude the proof, note that $k_t = x_t y_t$, so that

$$\frac{k_t^*}{A_t} = \frac{y_t^*}{A_t}\psi^{-1}\left(\frac{y_t^*}{A_t}\right) \equiv \tilde{G}^{-1}\left(\frac{y_t^*}{A_t}\right). \tag{8}$$

Inverting[7], we have

$$\frac{y_t^*}{A_t} = \tilde{G}\left(\frac{k_t^*}{A_t}\right) \tag{9}$$

and therefore

$$Y_t^* = A_t L_t \tilde{G}\left(\frac{K_t^*}{A_t L_t}\right) \equiv G(K_t^*, A_t L_t). \tag{10}$$

And this proves the key result: technical change is labor augmenting along the balanced growth path. Finally, since $Y_t^* = F(K_t^*, L_t; t) = G(K_t^*, A_t L_t)$, it must be the case that G satisfies the standard neoclassical properties as well.

4. DISCUSSION

Here's the one paragraph version of the proof. The crux is step 3 above. To begin, we notice that the familiar Cobb-Douglas property also holds more generally: the elasticity of output per worker with respect to the capital-output ratio is $\alpha(x;t)/(1-\alpha(x;t))$. Then, the fact that the capital share must be constant in steady state means that the production function must be factorable. That is, it must be possible, at least in steady state, to write the production function as $y_t = A(t)\psi(x)$. But this means that

[7] To show invertibility, differentiate:

$$\begin{aligned}
\tilde{G}^{-1}(z) &= z\psi^{-1}(z) \\
\Rightarrow \frac{d\tilde{G}^{-1}}{dz} &= \psi^{-1}(z) + z\frac{d\psi^{-1}(z)}{dz} \\
&= \psi^{-1} + \frac{z}{\left(\frac{d\psi}{dx}\right)} \\
&> 0 \quad \forall z > 0
\end{aligned}$$

as $\psi = y/A$ is always positive and $d\psi/dx$ is also always positive. So $\tilde{G}(\cdot)$ exists.

y_t/A_t and k_t/A_t must be constant as well, and one can really just look at the production function $y_t/A_t = F(k_t/A_t, 1/A_t; t)$ to see that this requires technical change to be labor augmenting.

This reasoning leads to the following intuition, which is the most suggestive way we have found to understand the labor-augmenting result. Divide both sides of the production function by output, yielding the "balance" expression $1 = F(K_t/Y_t, 1/y_t, t)$. In steady state, the capital-output ratio must be constant and y_t grows at a constant exponential rate. To satisfy the balance equation, then, technical change must exactly offset the fact that "effective labor" is falling at the rate of growth of y_t. That is, technical change must be labor augmenting.

This use of the word "balance" is suggestive in another way. In particular, consider the familiar phrase "balanced growth path" and ask what it is that is balanced along such a path. One is tempted to reply that it is the capital-output ratio that is somehow balanced, but a careful look at Uzawa's result suggests something different. To see this, consider as an example the special case of factor-augmenting technical change. Suppose $Y_t = F(B_tK_t, A_tL_t)$, where B_t and A_t grow exogenously at constant exponential rates. What must be "balanced" to get a steady state with positive and constant factor shares? The answer is the growth rates of the two effective inputs, B_tK_t and A_tL_t. If one effective input grows faster, a trend is induced in the factor shares unless the production function is Cobb-Douglas. Because capital accumulates and therefore inherits the trend in A_tL_t, the balance condition then implies that technical change must be purely labor augmenting.[8]

[8] To recall why, consider first the case where B_t is constant. In this case, it is well known that K_t and A_tL_t grow at the same rate in steady state. Now think about what would happen if B_t were to grow as well. Then B_tK_t would have to grow faster than A_tL_t. Growth would not be balanced and the factor shares would trend.

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