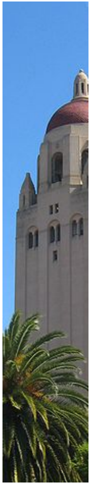

Recipes and Economic Growth: A Combinatorial March Down an Exponential Tail

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Combinatorics and Pareto

- Weitzman (1998) and Romer (1993) suggest combinatorics important for growth.
  - Ideas are combinations of ingredients
  - Combinations from a child's chemistry set $>$ # atoms in the universe
  - But absent from state-of-the-art growth models?
- Kortum (1997) and Gabaix (1999) on Pareto distributions
  - Kortum: Draw productivities from a distribution $\Rightarrow$ Pareto tail is essential
  - Gabaix: Pareto distribution (cities, firms, income) *results from* exponential growth

Do we really need the fundamental idea distribution to be Pareto?

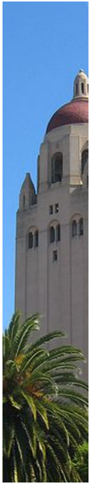
Two Contributions

- A simple but useful theorem about extreme values
  - The max extreme value depends on
    (1) the number of draws
    (2) the shape of the upper tail

- Combinatorics and growth theory
  - Combinatorial growth: Cookbook from N ingredients $\Rightarrow 2^N$ recipes, with N growing exponentially (population growth)

    Combinatorial growth with draws from thin-tailed distributions (e.g. the normal distribution) yields exponential growth

  - Pareto distributions are not required — draw faster from a thinner tail



Basic Foundations

Theorem (A Simple Extreme Value Result)

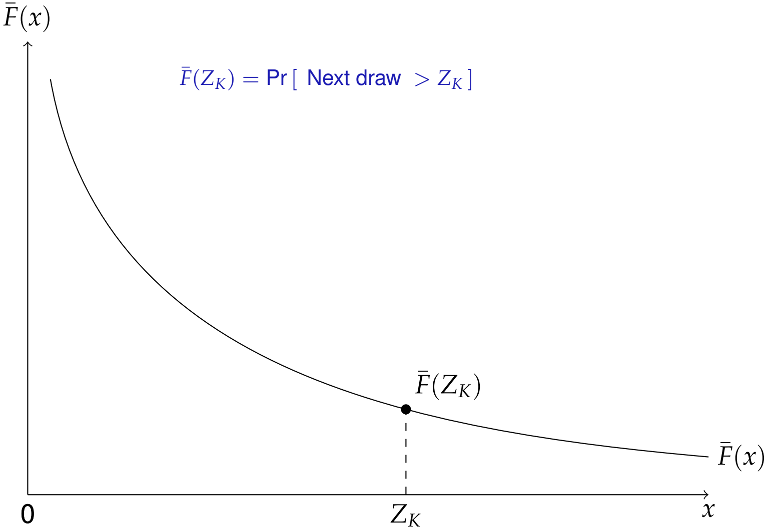
Let Z_K denote the maximum value from K i.i.d. draws from a continuous distribution $F(x)$, with $\bar{F}(x) \equiv 1 - F(x)$ strictly decreasing on its support. Then for $m \geq 0$

$$\lim_{K \to \infty} \Pr\left[\ K\bar{F}(Z_K) \geq m\ \right] = e^{-m}$$

As K increases, the max Z_K rises so as to stabilize $K\bar{F}(Z_K)$.

The shape of the tail of $\bar{F}(\cdot)$ and the way K increases determines the rise in Z_K

Graphically: Unpacking $K\bar{F}(Z_K)$

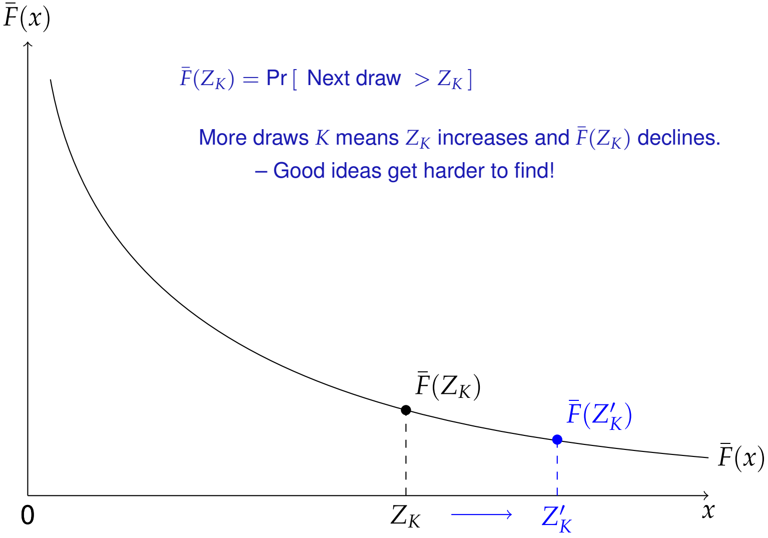


Graphically: Unpacking $K\bar{F}(Z_K)$

$\bar{F}(Z_K) = \text{Pr}\,[\text{ Next draw } > Z_K\,]$

More draws K means Z_K increases and $\bar{F}(Z_K)$ declines.

– Good ideas get harder to find!



Graphically: Unpacking $K\bar{F}(Z_K)$

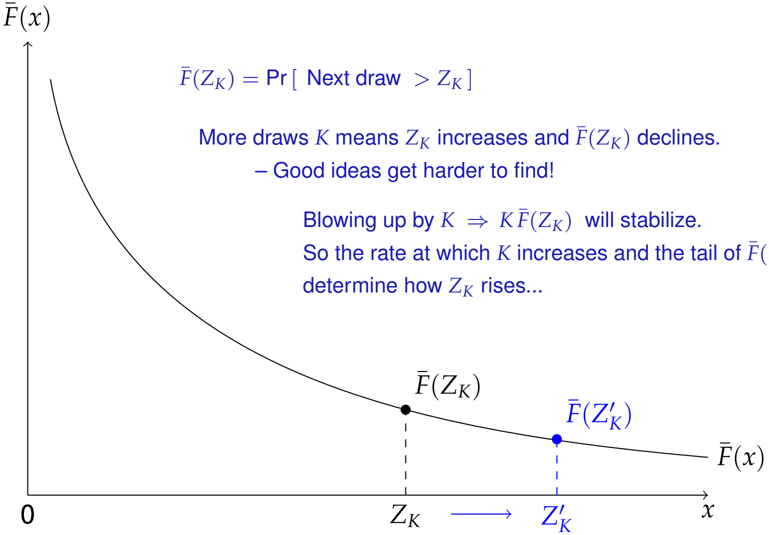
$\bar{F}(Z_K) =$ Pr [Next draw $> Z_K$]

More draws K means Z_K increases and $\bar{F}(Z_K)$ declines.

– Good ideas get harder to find!

Blowing up by $K \Rightarrow K\bar{F}(Z_K)$ will stabilize.

So the rate at which K increases and the tail of $\bar{F}(\cdot)$ determine how Z_K rises...



Intuition

$$K\bar{F}(Z_K) = \varepsilon + o_p(1)$$

$$\Rightarrow \quad \bar{F}(Z_K) \equiv \text{Pr}\left[\text{ Next draw } > Z_K\right] \sim \frac{1}{K}$$

- Theory of records: Suppose K i.i.d. draws for temperatures.
  - Unconditional probability that today is a new record high = $1/K$
  - This result is similar, but conditional instead of unconditional

$\Rightarrow$ $\bar{F}(Z_K)$ *falls like* $1/K$ *for any distribution!*

$\Rightarrow$ Z_K *rises like* $\bar{F}^{-1}(1/K)$

Proof of Theorem 1

- Given that Z_K is the max over K i.i.d. draws, we have

$$\begin{aligned}\Pr[Z_K \le x] &= \Pr[z_1 \le x, z_2 \le x, \ldots, z_K \le x] \\ &= (1 - \bar{F}(x))^K\end{aligned}$$

- Let $M_K \equiv K\bar{F}(Z_K)$ denote a new random variable. Then for $0 < m < K$

$$\begin{aligned}\Pr[M_K \ge m] &= \Pr\left[K\bar{F}(Z_K) \ge m\right] \\ &= \Pr\left[\bar{F}(Z_K) \ge \frac{m}{K}\right] \\ &= \Pr\left[Z_K \le \bar{F}^{-1}\left(\frac{m}{K}\right)\right] \\ &= \left(1 - \frac{m}{K}\right)^K \to e^{-m} \quad \text{QED.}\end{aligned}$$

Remarks

- Simpler and different from the standard EVT
  - If $\frac{Z_K - b_K}{a_K}$ converges in distribution, then it converges to one of three types
  - Which one depends on the tail properties of $F(\cdot)$
- We will see later that Theorem 1 covers cases not covered by EVT
- Intuition for why so few conditions on $F(\cdot)$ are required:
  - For any distribution of x, $\bar{F}(x)$ is Uniform[0,1]
  - Min over K draws from a uniform, scaled up by K, is exponential = $K\bar{F}(Z_K)$ (from standard EVT)
  - Barton and David (1959), Galambos (1978, Chapter 4), and Embrechts et al (1997, Prop 3.1.1) have related results

Example: Kortum (1997)

- Pareto: $\bar{F}(x) = x^{-\beta}$
- Apply Theorem 1:

$$\begin{aligned} K\bar{F}(Z_K) &= \varepsilon + o_p(1) \\ KZ_K^{-\beta} &= \varepsilon + o_p(1) \\ \frac{K}{Z_K^{\beta}} &= \varepsilon + o_p(1) \\ \frac{Z_K}{K^{1/\beta}} &= (\varepsilon + o_p(1))^{-1/\beta} \end{aligned}$$

- Exponential growth in K leads to exponential growth in Z_K

$$g_Z = g_K/\beta$$

β = how thin is the tail = rate at which ideas become harder to find

Example: Drawing from an Exponential Distribution

- Exponential: $\bar{F}(x) = e^{-\theta x}$

$$
\begin{aligned}
K\bar{F}(Z_K) &= \varepsilon + o_p(1) \\
Ke^{-\theta Z_K} &= \varepsilon + o_p(1) \\
\Rightarrow \quad \log K - \theta Z_K &= \log(\varepsilon + o_p(1)) \\
\Rightarrow \quad Z_K &= \frac{1}{\theta}[\log K - \log(\varepsilon + o_p(1))] \\
\Rightarrow \quad \frac{Z_K}{\log K} &= \frac{1}{\theta}\left(1 - \frac{\log(\varepsilon + o_p(1))}{\log K}\right)
\end{aligned}
$$

$$\frac{Z_K}{\log K} \xrightarrow{p} \text{Constant}$$

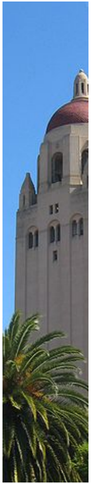
Drawing from an Exponential (continued)

$$\frac{Z_K}{\log K} \xrightarrow{p} \text{Constant}$$

- Z_K grows with $\log K$
  - If K grows exponentially, then Z_K grows linearly
- Definition of **combinatorial growth:** $K_t = 2^{N_t}$ with $N_t = N_0 e^{g_N t}$

$$g_Z = g_{\log K} = g_N$$

Combinatorial growth with draws from a thin-tailed distribution delivers exponential growth!



Growth Model

Setup

- Cookbook is a collection of K_t recipes
- At a point in time, researchers have evaluated all recipes from N_t ingredients
  - Each ingredient can either be included or excluded, so $K_t = 2^{N_t}$ (which equals $\sum_{k=0}^{N_t} \binom{N_t}{k}$, the sum of all combinations)
- Research = learning the "productivity" of the new recipes that come from adding a new ingredient
- $\dot{N}_t = \alpha R_t \Rightarrow$ each researcher can evaluate α new ingredients each period
  - R_t grows with population $\Rightarrow$ so does N_t

Combinatorial growth: Cookbook of $K = 2^N$ recipes from N ingredients, with N growing exponentially

Corollary (Poisson version of Theorem 1)

Let Z_K denote the maximum over P independent draws from a distribution with a strictly decreasing and continuous tail cdf $\bar{F}(x)$ and suppose P is distributed as Poisson with parameter K. Then for $0 < m < K$

$$\Pr\left[\, K\bar{F}(Z_K) \geq m \,\right] = \frac{e^{-m} - e^{-K}}{1 - e^{-K}}.$$

- Applies at each point in time, not just asymptotically
- Integrate across a continuum of sectors to make aggregate growth deterministic
- (Thanks to Sam Kortum for this suggestion)

Economic Environment: Like Kortum (1997) but with Weibull/Combinatorial Growth

Aggregate output	$Y_t = \left(\int_0^1 Y_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$ with $\sigma > 1$
Variety i output	$Y_{it} = Z_{Kit}\left(M_{it}^{-\frac{1}{\rho}} \sum_{j=1}^{M_{it}} x_{ijt}^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}}$ with $\rho > 1$
Production of ingredients	$x_{ijt} = L_{ijt}$
Best recipe	$Z_{Kit} = \max_{c=1,\ldots,\tilde{K}_{it}} z_{ic}$ where $\tilde{K}_{it} \sim$ Poisson(K_t)
Weibull distribution of z_{ic}	$z_{ic} \sim F(x) = 1 - e^{-x^{\beta}}$ β = how thin is tail
Number of ingredients evaluated	$\dot{N}_t = \alpha R_t^{\lambda} N_t^{\phi}, \quad \phi < 1$
Cookbook (Poisson parameter)	$K_t = 2^{N_t}$
Resource constraint: workers	$L_{it} = \sum_{j=1}^{M_{it}} L_{ijt}$ and $\int_0^1 L_{it} di = L_{yt}$
Resource constraint: R&D	$R_t + L_{yt} = L_t$
Population growth (exogenous)	$L_t = L_0 e^{g_L t}$

Allocation

- Consider the allocation of labor that maximizes Y_t at each date with a constant fraction of people working in research
  - L_{ijt} maximizes Y_t
  - $R_t = \bar{s}L_t$

- Number of ingredients evaluated (eventually) grows at a constant rate

$$\frac{\dot{N}_t}{N_t} = \frac{R_t^\lambda}{N_t^{1-\phi}} \quad \Rightarrow \quad g_N = \frac{\lambda g_L}{1-\phi}$$

- And we have combinatorial growth in the number of recipes in the cookbook

$$K_t = 2^{N_t} \quad \Rightarrow \quad g_{\log K} = g_N$$

Applying Theorem 1 to the Weibull Distribution

- Suppose $y \sim$ Exponential. Let $y \equiv x^\beta$. Then $x \sim$ Weibull: $\bar{F}(x) = e^{-x^\beta}$

$$\frac{\max y}{\log K} \xrightarrow{p} \text{Constant}$$

$$\Rightarrow \quad \frac{\max x^\beta}{\log K} \xrightarrow{p} \text{Constant}$$

$$\Rightarrow \quad \frac{\max x}{(\log K)^{1/\beta}} \xrightarrow{p} \text{Constant}$$

- Therefore

$$\boxed{g_{Z_K} = \frac{g_{\log K}}{\beta} = \frac{g_N}{\beta} = \frac{1}{\beta}\frac{\lambda g_L}{1-\phi}}$$

(slightly more complicated with Poisson process, but same idea)

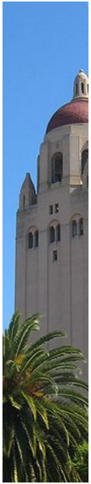
Remarks

$$g_{Z_K} = \frac{g_{\log K}}{\beta} = \frac{g_N}{\beta} = \frac{1}{\beta}\frac{\lambda g_L}{1-\phi}$$

- This is the growth rate of output per person in the growth model
- Combinatorial march down a Weibull tail
- Growth rate depends on
  - Population growth = growth rate of researchers
  - λ and ϕ = how researchers evaluate ingredients
  - Allows $\phi > 0$: it may get easier (or harder) to evaluate ingredients
  - While β captures the degree to which good ideas get harder to find

Can the distribution shift out over time?

- Consider all the technologies that could ever be invented. They are recipes.
  - Let $\bar{F}(x)$ be the associated distribution of productivities
  - That doesn't shift...
- What's behind the question: some technologies cannot be invented before others
  - The smartphone could not come *before* electricity, radio, and semiconductors
- Answer: Suppose new ideas are future ingredients
  - Ingredients must be evaluated in a specific order
  - Nothing changes...



Generality?

For what distributions do combinatorial draws $\Rightarrow$ exponential growth?

Theorem (A general condition for combinatorial growth)

Consider the growth model above but with $z_i \sim F(z)$ *as a general continuous and unbounded distribution, where* $F(\cdot)$ *is monotone and differentiable. Let* $\eta(x)$ *denote the elasticity of the tail cdf* $\bar{F}(x)$*; that is,* $\eta(x) \equiv -\frac{d \log \bar{F}(x)}{d \log x}$*. Then*

$$\lim_{t \to \infty} \frac{\dot{Z}_{Kt}}{Z_{Kt}} = \frac{g_N}{\alpha}$$

if and only if

$$\lim_{x \to \infty} \frac{\eta(x)}{x^{\alpha}} = \text{Constant} > 0$$

for some $\alpha > 0$*.*

Remarks

$$\frac{\dot{Z}_{Kt}}{Z_{Kt}} \to \frac{g_N}{\alpha} \iff \lim_{x\to\infty} \frac{\eta(x)}{x^\alpha} = \text{Constant} > 0$$

- Thinner tails require faster draws but still require power functions:
  - It's just that the elasticity itself is now a power function!
- Examples
  - Weibull: $\bar{F}(x) = e^{-x^\beta} \Rightarrow \eta(x) = x^\beta$
  - Normal: $\bar{F}(x) = 1 - \int_{-\infty}^{x} e^{-u^2/2} du \Rightarrow \eta(x) \sim x^2$ – like Weibull with $\beta = 2$
- Intuition
  - Kortum (1997): $\bar{F}(x) = x^{-\beta} \Rightarrow \eta(x) = \beta$ so $K_t = e^{nt}$ is enough
  - Here: $\bar{F}(x) = e^{-x^\beta}$ so must march down tail exponentially faster, $K_t = 2^{e^{nt}}$

For what distributions do combinatorial draws ⇒ exponential growth?

- Combinatorial draws lead to exponential growth for many familiar distributions:
  - Normal, Exponential, Weibull, Gumbel
  - Gamma, Logistic, Benktander Type I and Type II
  - Generalized Weibull: $\bar{F}(x) = x^{\alpha} e^{-x^{\beta}}$ or $\bar{F}(x) = e^{-(x^{\beta}+x^{\alpha})}$
  - Tail is dominated by "exponential of a power function"
- When does it not work?
  - lognormal: If it works for normal, then $\log x \sim$ Normal means percentage increments are normal, so tail will be too thick!
  - logexponential = Pareto
  - Surprise: Does *not* work for all distributions in the Gumbel domain of attraction (not parallel to Kortum/Frechet).

Scaling of Z_K for Various Distributions

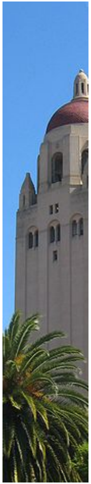
Distribution	cdf	Z_K behaves like	Growth rate of Z_K for $K = 2^N$
Exponential	$1 - e^{-\theta x}$	$\log K$	g_N
Gumbel	$e^{-e^{-x}}$	$\log K$	g_N
Weibull	$1 - e^{-x^\beta}$	$(\log K)^{1/\beta}$	$\frac{g_N}{\beta}$
Normal	$\frac{1}{\sqrt{2\pi}} \int e^{-x^2/2} dx$	$(\log K)^{1/2}$	$\frac{g_N}{2}$
Lognormal	$\frac{1}{\sqrt{2\pi}} \int e^{-(\log x)^2/2} dx$	$\exp(\sqrt{\log K})$	$\frac{g_N}{2} \cdot \sqrt{N}$
Gompertz	$1 - \exp(-(e^{\beta x} - 1))$	$\frac{1}{\beta} \log(\log K)$	Arithmetic
Log-Pareto	$1 - \frac{1}{(\log x)^\alpha}$	$\exp(K^{1/\alpha})$	Romer!

Microfoundations for Romer (1990)

- Kortum (1997) found there is no process satisfying EVT that delivers Romer (1990) result of exponential growth with constant flow of draws

- But Theorem 1 shows us how to get it:
  - If $x \sim$ logpareto with $\alpha = 1$, then linear growth in K (e.g. $\dot{K}_t = \bar{L}$) gives exponential growth in max

- Implies $Z_K = \exp\left(K^{1/\alpha}(\tilde{\varepsilon} + o_p(1)\right)$
  - No affine transformation of Z_K works, which is why EVT fails (need to take logs)
  - Implies that log productivity is Fréchet in cross-section
    - much thicker tail than we observe in the data
    - variance of log productivity would rise over time



Evidence from Patents

Combinatorial growth matches the patent data

Rate of Innovation?

- Kortum (1997) was designed to match a key “fact”: that the flow of patents was stationary
  - Never clear this fact was true (see below)

- Flow of patents in the model?
  - Theory of record-breaking: $p(K) = 1/K$ is the fraction of ideas that exceed the frontier [cf Theorem 1: $\bar{F}(Z_K) = \frac{1}{K}(\varepsilon + o_p(1))$]
  - Since there are $\dot{K}$ recipes added to the cookbook every instant, the flow of patents is

$$p(K)\dot{K} = \frac{\dot{K}_t}{K_t}$$

  - This is constant in Kortum (1997) $\Rightarrow$ constant flow of patents

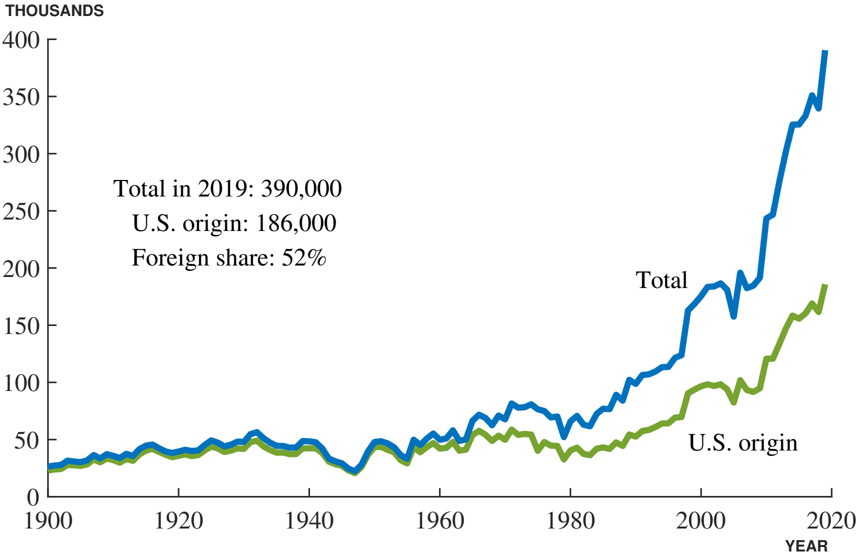
Flow of Patents in Combinatorial Growth Model?

- Simple case: $\dot{N}_t = \alpha R_t$ (e.g. $\lambda = 1$ and $\phi = 0$ in $\dot{N}_t = \alpha R_t^{\lambda} N_t^{\phi}$)
- Then

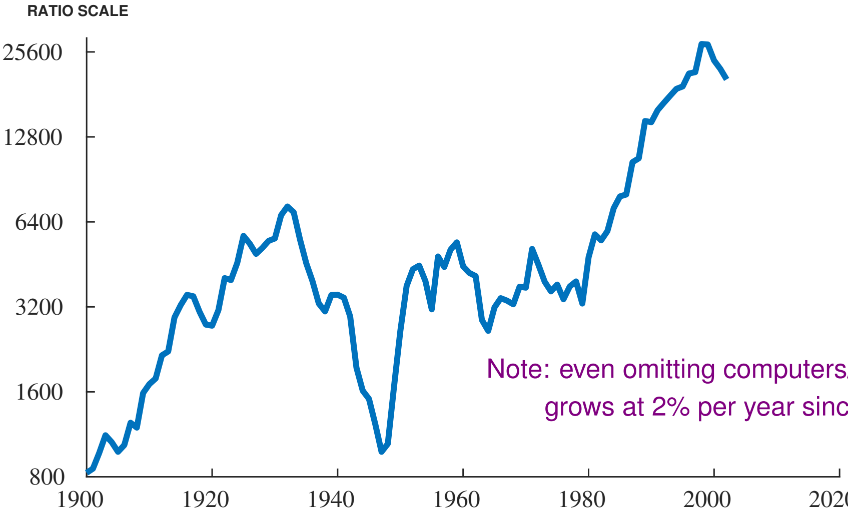
$$
\begin{aligned}
K_t &= 2^{N_t} \\
\Rightarrow \frac{\dot{K}_t}{K_t} &= \log 2 \cdot \dot{N}_t \\
&= \log 2 \cdot \alpha R_t \\
&= \log 2 \cdot \alpha \bar{s} L_0 e^{g_L t}
\end{aligned}
$$

- That is, the combinatorial growth model predicts that the number of new patents should grow exponentially over time
  - When ideas are small, it takes a growing number to generate exponential growth

Annual Patent Grants by the U.S. Patent and Trademark Office

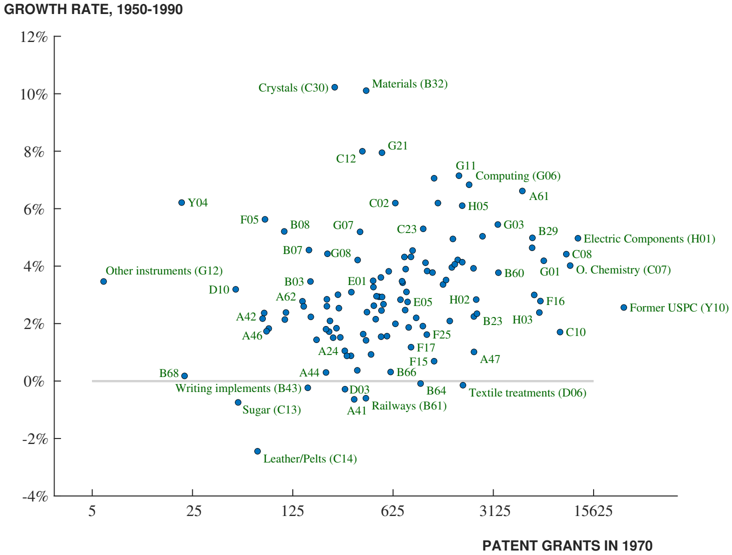


Breakthrough Patents from Kelly et al (2021)

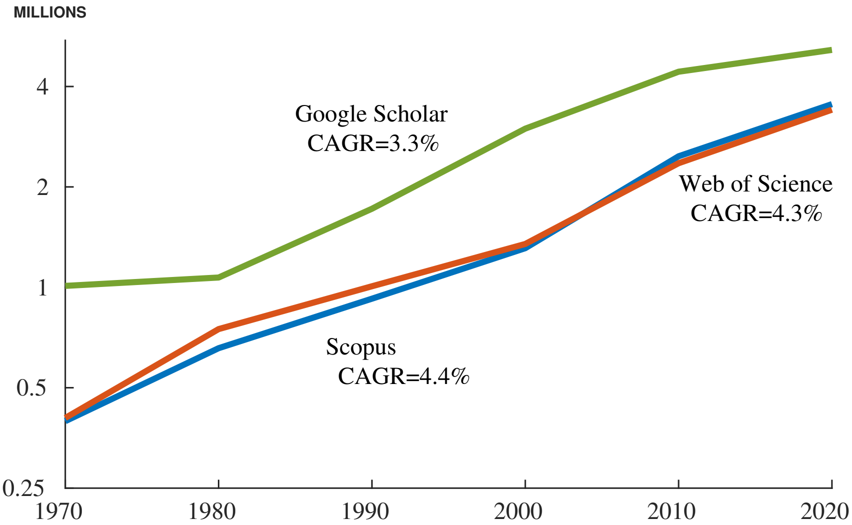


Note: even omitting computers/electronics, grows at 2% per year since 1950

U.S. Patent Growth by Technology Class, 1950–1990



Annual Academic Publication Counts, 1970–2020



Remarks

- In Kortum (1997), rise in patents should correspond to a rise in growth rates.
  - Data seem more consistent with the combinatorial growth model
  - (Important caveat: meaning of a “patent” is not stable over time)
- Can researchers evaluate a combinatorially growing list of recipes?
  - Maybe it is only the “good” ideas that take time
  - With $\lambda = 1$ and $\phi = 0$, the number of good ideas per researcher is constant
  - Chess players find the best line from an exploding set of possibilities
  - Henri Poincare quote

Implications for Future Research

- Wherever Pareto has been assumed in the literature, perhaps we can use thin tails?
  - Technology diffusion, trade, search, productivity
- Beautiful feature of Kortum (1997)
  - Pareto assumption $\Rightarrow$ theory of growth, markups, and firm heterogeneity
- If ideas are "small," we lose these connections
  - Combinatorial theory of growth
  - But markups and heterogeneity disappear asymptotically
  - Gaps between ideas are too small to provide this theory
- Opportunity! Need new theory of markups and heterogeneity