Set-Membership Identification of Systems with Parametric and Nonparametric Uncertainty

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Abstract—**A method is presented for parameter set estimation where the system model is assumed to contain both parametric and nonparametric uncertainty. In the disturbance-free case, the parameter set estimate is guaranteed to contain the parameter set of the true plant. In the presence of stochastic disturbances, the parameter set estimate obtained from finite data records is shown to have the property that it contains the true-plant parameter set with probability one as the data length tends to infinity.**

I. Introduction

In the traditional adaptive control system, the identified model is used for on-line controller design without any regard for errors between this model and the true system which generated the data. The identified model is usually selected out of a model set with unknown parameters as depicted in Fig. 1. The controller is designed as if the parameter estimates were in fact the correct parameters for describing the plant. This is known as applying the certainty equivalence principle. In the ideal case, it is assumed that there exist parameters, which if known, would precisely account for the measured data. Even in this ideal case, the transient errors between the identified model and the true system can be so large as to completely disrupt the performance. In the usual (nonideal) case, the true system is not in the model set, therefore, both unacceptable transient or asymptotic behavior can occur, e.g., [1].

Following the ancient Greek adage,[1] "*Well begun, half done*," one ought to construct, at the outset, an adaptive control system which specifically accounts for the inevitable model error, i.e., an adaptive *robust* control. Depicted in Fig. 2 is our proposed scheme where the traditional parameter estimator is replaced with an estimator that produces a model set. Thus, point estimation of a single model is replaced with *set-membership identification*. The estimated model set can contain both parametric and nonparametric descriptions of uncertainty arrived at from both measured and prior data.

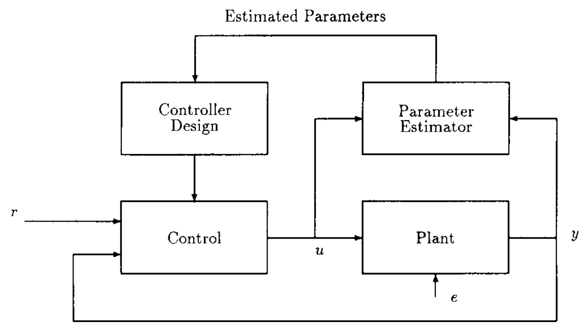

Fig. 1. Traditional adaptive control system with parameter estimator.

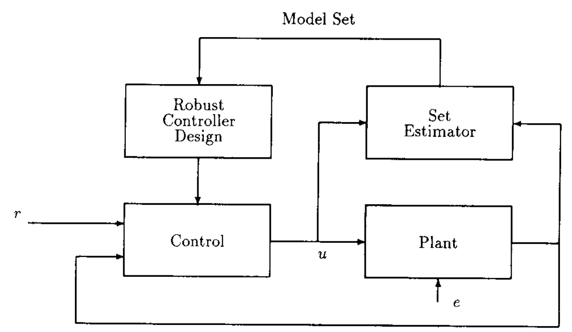

Fig. 2. Adaptive control with set estimator.

We also replace the traditional controller design algorithm with a robust controller design algorithm which accepts the model set format. By referring to a robust controller we mean a controller that achieves some specific set of specifications for any plant model in the model set. The robust controller design thus takes a set of models as input and produces a controller that is guaranteed to meet the specifications for all models in this set. The robust controller design can also report the worst-case performance with respect to the model set. It is also true that if the model set is too large, or the specifications are too tight, then no robust controller will exist.

During the transient or learning phase, the estimated model set could be a poor representation of the true system as it could be quite large. However, if the system which generated the measured data is contained in the estimated set, the robust controller will be stabilizing, though may be of low author-

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[1] Αρξη′ ημισυ παgnτων. Literally translated: (The) beginning (is the) half of all [16].

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ity. Conversely, if the model set becomes smaller after some time, this will be reflected in a higher authority controller with more desirable performance characteristics.

It is important to point out, and even emphasize, that although this approach is inspired by a separation principle, *it is not optimal*. Roughly speaking, set estimation and robust controller design might benefit from being coupled. For example, the input u might be temporarily manipulated in such a way so that the set estimator could rapidly learn and therefore improve future performance at the expense of current performance. In a purely Bayesian framework, notions of optimality along this line are made precise in [9].

Although not guaranteed to be optimal, the scheme shown in Fig. 2 is at least less heuristic than the traditional scheme of Fig. 1. For example, if the set estimator is *consistent*, that is, the true plant is in the estimated model set, and moreover, if we stop adapting at any given point, then we are guaranteed a worst-case performance as reported by the robust controller design.

In this paper, we address the problem of parameter set estimation where the system model contains both parametric and nonparametric uncertainty. In our formulation, we use the measured data to delineate a parametric set which accounts for *a priori* knowledge of nonparametric dynamics and disturbances. Observe that if measured data is not used, then the identified model set consists of a constant model set and the "adaptive" controller reduces to a single robust design. We can also recover the traditional adaptive scheme by replacing the robust design with a heuristic design which uses a typical model in the set, e.g., the "center" or "average" model.

We will not address the robust control design issues as different methodologies for robust control design, particularly for plants with uncertain nonparametric linear dynamics, can be found in [26], [8], and [12]. Methods for robust control design of plants with parametric uncertainty are described in [2], [5] and the references therein. In the case of parametric set-membership uncertainty, minimax controllers are considered in [22] and [21].

At present, there are several competing and complementary methodologies for the design of set estimators, e.g., [29], [20], [17], [14], [18], and [32]. Related work on the limitations of identification of linear-time invariant systems can be found in [13], [15], [24], and [28]. Our work here follows closely to that described in [31], [32], and [18] for the disturbance-free case with nonparametric uncertainty, and in [23] for the disturbance case. The parameter sets developed here are similar in form to those developed in [10], [11], [25], and [3] for the case with no nonparametric uncertainty but with bounded disturbances. The foundation and impetus for much of the work in parameter set-membership identification can be traced back to [27], and [4] for the state-estimation problem.

The paper is organized as follows. After introducing some notation and standard definitions in the next section, the problem is formulated in Section III. Parameter set estimates for the disturbance-free equation-error case are developed in Section IV. In the presence of stochastic disturbances, equation-error parameter set estimates computable from finite data records are presented in Section V. Extensions to the output-error case and deterministic disturbances are discussed in Section VI. The paper concludes with some remarks in Section VII.

II. Notation and Preliminaries

Transfer Functions: In this paper, we consider sampled-data systems with transfer functions in the complex variable z. If the system is denoted by G, then its transfer function is denoted by $G(z)$. Typically, $G(z)$ is obtained as the zero-order hold equivalent of a continuous-time transfer function $P(s)$. Thus,

$$G(z) = \mathscr{ZOH}\{P(s)\} \tag{1}$$

$$\triangleq (1 - z^{-1})\mathscr{Z}\left\{\frac{1}{s}P(s)\right\} \tag{2}$$

where $\mathscr{ZOH}\{\cdot\}$ and $\mathscr{Z}\{\cdot\}$ denote the zero-order hold and the usual z-transform operations, respectively.

A transfer function $G(z)$ is *stable* if all its poles are strictly inside the unit circle $|z| = 1$. The frequency response of $G(z)$ is the function $G(e^{j\omega})$ restricted to the domain $|\omega| \leq \pi$, where ω is the frequency variable normalized with respect to the sampling frequency. For a stable transfer function $G(z)$, the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ norms are defined as

$$\|G\|_{\mathscr{H}_\infty} \triangleq \sup_{|\omega| \leq \pi} |G(e^{j\omega})| \tag{3}$$

$$\|G\|_{\mathscr{H}_2} \triangleq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2 \, d\omega\right)^{1/2}. \tag{4}$$

Sequences: A sequence x is a function of discrete-time points, i.e., $x: \mathbb{N} \to \mathbb{R}^p$ where $\mathbb{N} = \{1, 2, \cdots\}$ is the set of positive integers. We write $x(t)$ to mean the value of the sequence at a particular time t, normalized with respect to the sampling interval. Hence, time takes on integer values with initial time defined as $t = 1$.

Following [24], a sequence x is *quasi-stationary* if $\mathscr{E}(x(t))$ is bounded for all t and its autocorrelation

$$r_{xx}(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathscr{E}(x(t)x(t-\tau)) \tag{5}$$

exists for all integers τ, where $\mathscr{E}(\cdot)$ denotes the expectation operator. If x is a deterministic sequence, the expectation is without effect and quasi-stationary then means that x is a bounded sequence such that the limits

$$r_{xx}(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} x(t)x(t-\tau) \tag{6}$$

exist. For easy notation, we introduce the symbol $\bar{\mathscr{E}}$ by

$$\bar{\mathscr{E}}(x) \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathscr{E}(x(t)). \tag{7}$$

The power spectrum of x is defined as

$$S_{xx}(\omega) \triangleq \sum_{\tau=-\infty}^{\infty} r_{xx}(\tau)e^{-j\omega\tau}. \tag{8}$$

This leads to the power in x given by

$$r_{xx}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega)\,d\omega = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\mathscr{E}\left(x(t)^2\right). \tag{9}$$

Similar definitions apply to the cross spectrum $S_{xy}(\omega)$ of the sequences x and y.

The sample-mean operator $\mathscr{E}_k(\cdot)$ is defined to be

$$\mathscr{E}_k(x) \triangleq \frac{1}{k}\sum_{t=1}^{k} x(t). \tag{10}$$

We use $\|x\|_{k2}$ to denote the truncated l_2-norm of a sequence

$$\|x\|_{k2} \triangleq \left(\sum_{t=1}^{k} x(t)^2\right)^{\frac{1}{2}} \tag{11}$$

hence,

$$\mathscr{E}_k(x^2) = \frac{1}{k}\|x\|_{k2}^2. \tag{12}$$

Linear Operators: The notation Gx means the sequence obtained when the system G operates on the sequence x. We write $(Gx)(t)$ to mean the value at time t of the sequence Gx.

When we say that G is a linear-time invariant system, we mean that Gx is the convolution operation

$$(Gx)(t) = \sum_{k=0}^{t-1} g(k)x(t-k) \tag{13}$$

where the sequence g is the pulse response of G. Thus, G has the transfer function

$$G(z) = \sum_{k=0}^{\infty} g(k)z^{-k}. \tag{14}$$

The above definition restricts the sequence Gx to $t \geq 1$. Hence, the system G can be regarded as having no memory of events prior to $t = 1$, the initial time. Roughly, this means all initial conditions are zero.

To reduce notation, we use the transform variable z to denote the shift operator, so $z^k x(t) = x(t+k)$, $z^{-k}x(t) = x(t-k)$, and $z^k x$ shifts each member of the sequence x.

III. Problem Formulation

The problem is to use the measured sampled data

$$\{y, u: t = 1, \cdots, N\} \tag{15}$$

to identify a model set suitable for robust control design. The system which produced the data is assumed to be a linear-time invariant system of the form

$$y = Gu + v \tag{16}$$

where G is a linear-time invariant system with transfer function $G(z)$, u is an applied input, y is the measured output, and v is a disturbance as seen at the output. It is also assumed that both y and u have finite power, that is, $r_{yy}(0) < \infty$ and $r_{uu}(0) < \infty$.[2]

A. Model Set Assumptions

The model set $\mathscr{M}$ is defined as follows:

$$\mathscr{M} \triangleq \{y = Gu + v: G \in \mathscr{G}, v \in \mathscr{V}\} \tag{17}$$

where $\mathscr{G}$ is the set of linear-time invariant systems and $\mathscr{V}$ is the set of disturbances. It is assumed that the true system (16) is a member of the model set $\mathscr{M}$. The reader should be cautioned that G defined in the model set $\mathscr{M}$ is *not* the same as G in (16). To avoid adding more subscripts G_{true}, etc., unless otherwise stated as part of some set, e.g., $G \in \mathscr{G}$, the symbols G, y, u, and v refer to the true system (16).

We first concentrate on the disturbance-free case, i.e., $v = 0$, in the next section. The disturbance set $\mathscr{V}$ is discussed later in Section V.

The set of linear-time invariant systems is defined by

$$\mathscr{G} \triangleq \{G_\theta(1 + \Delta_G W_G): \theta \in \Theta_{\text{prior}}, \|\Delta_G\|_{\mathscr{H}_\infty} \leq 1\} \tag{18}$$

where $G_\theta(z)$ is a parametric transfer function with parameters $\theta \in \Theta_{\text{prior}}$, referred to as the *prior parameter set*. The system $\Delta_G W_G$ is referred to as the *multiplicative nonparametric uncertainty*. It is a dynamic uncertainty characterized by an uncertain but unity bounded stable-transfer function $\Delta_G(z)$ and a known stable-transfer function $W_G(z)$. Note that $W_G(z)$ acts as a frequency weighting function, whose frequency response magnitude $|W_G(e^{j\omega})|$ reflects the size of the nonparametric uncertainty. Since a parametric model of a system is never complete unless we have some idea on its limitations and accuracies, we assume that the uncertainty weighting function $W_G(z)$ is known. Having knowledge of W_G is precisely the assumption made in robust control design, e.g., [8]. However, the center of the model set is fixed in robust control design, here it is parametric, i.e., G_θ.

Suppose the true system G is in $\mathscr{G}$ and we are interested in all the possible representations of G in $\mathscr{G}$. Solving for Δ_G in (18) in terms of G and θ, we get

$$\Delta_G = \frac{G - G_\theta}{W_G G_\theta}. \tag{19}$$

We define

$$\Theta^* \triangleq \left\{\theta: \left\|\frac{G - G_\theta}{W_G G_\theta}\right\|_{\mathscr{H}_\infty} \leq 1\right\} \tag{20}$$

and refer to Θ^* as the *parametric limit set* because it does not depend on the data set but rather on the true but unknown system G. As a result, $\Theta^* \cap \Theta_{\text{prior}}$ is the set of all possible parameter values consistent with the assumption that the true system G is in $\mathscr{G}$. Consequently, it is not possible to consider a "true" parameter value because any member of

[2] Input and output sequences with finite power occur, for example, when G is stable and u has finite power, or when G, not necessarily stable is stabilized by an appropriate feedback and the exogeneous inputs to the feedback system have finite power.

$\Theta^* \cap \Theta_{\text{prior}}$ is a possibility since the decomposition of G into G_θ and Δ_G is not unique. Thus, the goal is to obtain an estimate of the set Θ^* from the measured data.

Throughout the remainder of the paper we further characterize the parametric transfer function $G_\theta(z)$ by using the standard ARX form [24]

$$\begin{aligned} G_\theta(z) &= B_\theta(z)/A_\theta(z) \\ B_\theta(z) &= b_1 z^{-1} + \cdots + b_m z^{-m} \\ A_\theta(z) &= 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \\ \theta &= [a_1 \cdots a_n \, b_1 \cdots b_m]^T. \end{aligned} \tag{21}$$

Thus, the parameters are the coefficients in the parametric transfer function. With this parametrization, the limit set becomes

$$\Theta^* = \left\{ \theta : \left\| \frac{A_\theta G - B_\theta}{W_G B_\theta} \right\|_{\mathscr{H}_\infty} \leq 1 \right\}. \tag{22}$$

The problem we are addressing in this paper is to find an estimate of Θ^*. We should also point out that other than what is assumed for the transfer function $\Delta_G(z)$, we do not estimate it from the data. We first give an example of Θ^*, and then in the next section, describe a set estimator in the disturbance-free case.

B. Example of Limit Set

Suppose that the true transfer function is

$$G(z) = \mathscr{Z}\mathscr{O}\mathscr{H}\left\{ \left(\frac{10}{s+1} \right) \left(\frac{10^2}{s^2 + 2(0.005)(10)s + 10^2} \right) \right\}. \tag{23}$$

The sampling frequency is chosen to be $2\pi(10)$ rad/s or 10 Hz. Observe that the system has a simple pole at 1 rad/s, and a very lightly damped resonance at 10 rad/s. Suppose we are interested in obtaining a good low-frequency model by neglecting the resonance, but accounting for it as one realization of some nonparametric dynamics. Thus, select the parametric transfer function as

$$G_\theta(z) = \frac{bz^{-1}}{1 + az^{-1}}, \; \theta = \begin{bmatrix} a \\ b \end{bmatrix}. \tag{24}$$

Consider the following weights:

$$W_{G,1}(z) = 65\left[\mathscr{Z}\mathscr{O}\mathscr{H}\left\{ \frac{s+1}{s+5} \right\} \right]^4 \tag{25}$$

$$W_{G,2}(z) = W_{G,1}(z) - 65\left(\frac{1}{5} \right)^4. \tag{26}$$

Either of these weights can account for the resonance, but they reflect different prior low-frequency uncertainties. The weight $W_{G,1}$ reflects a low-frequency multiplicative uncertainty of about 10% where it has a dc gain of about 0.1, and it anticipates a rather large resonance at frequencies beyond about 10 rad/s where the magnitude of $W_{G,1}$ is greater than 100. $W_{G,2}$ is essentially the same but has a zero dc gain. Shown in Fig. 3 are the frequency response magnitudes and the multiplicative error with respect to a "nominal" parametric transfer function

$$G_{\theta_{\text{nom}}}(z) = \mathscr{Z}\mathscr{O}\mathscr{H}\left\{ \frac{10}{s+1} \right\}. \tag{27}$$

With the sampling frequency of 10 Hz,

$$\theta_{\text{nom}} = \begin{bmatrix} a_{\text{nom}} \\ b_{\text{nom}} \end{bmatrix} = \begin{bmatrix} -0.9048 \\ 0.9516 \end{bmatrix}. \tag{28}$$

This transfer function can be viewed as an approximation of $G(z)$ obtained by neglecting the resonance in (23). Remember, there is no true parameter value, rather, there is a true set Θ^*, one element of which is this nominal parameter value.

Points in the limit set corresponding to the above weights are shown in Fig. 4. These points are obtained by testing θ in (19) over a set of points. If a point's corresponding Δ_G satisfies $\|\Delta_G\|_{\mathscr{H}_\infty} \leq 1$, then it belongs to Θ^*. Since $W_{G,2}(e^{j\omega})$ is zero at $\omega = 0$, i.e., the dc gain of $G(z)$ is assumed known, and the two parameters in θ are constrained to lie on a line in the parameter space. The line becomes "blurred" in the limit set corresponding to $W_{G,1}$ because there is no frequency where the frequency response of $W_{G,1}$ is identically zero.

IV. Disturbance-Free Equation-Error Set Estimation

In the disturbance-free case, we have $v = 0$. Thus, the model set in (17) reduces to

$$\mathscr{M} = \{ y = Gu : G \in \mathscr{G} \} \tag{29}$$

with $\mathscr{G}$ given by (18).

Theorem 1: Suppose the measured data $\{y, u: \; t = 1, \cdots, N\}$ is generated from $y = Gu$ with $G \in \mathscr{G}$. Then the following holds:

$$\Theta^* \subseteq \Theta[N] \subseteq \Theta_k, \qquad \forall k \in [1, N], \forall N \in \mathbb{N} \tag{30}$$

where $\Theta[N]$ and Θ_k are given by

$$\Theta_k \triangleq \{ \theta : \| A_\theta y - B_\theta u \|_{k2} \leq \| W_G B_\theta u \|_{k2} \} \tag{31}$$

$$\Theta[N] \triangleq \bigcap_{k=1}^{N} \Theta_k. \tag{32}$$

Remarks: We refer to Θ_k or $\Theta[N]$ as *equation-error parameter sets* because the equation-error term $A_\theta y - B_\theta u$ appears in the definition [24]. Observe that the equation-error sets depend only on the measured data and the known bounding transfer function $W_G(z)$. Because Θ^* is a subset, it follows that Θ_k for any $k \in [1, N]$ or $\Theta[N]$ is an estimate of Θ^*. These sets are easy to compute as will be shown in Section IV-C. First we prove the theorem.

Proof: First, recall the following fact from [7]. If T is a stable linear-time invariant operator with transfer function

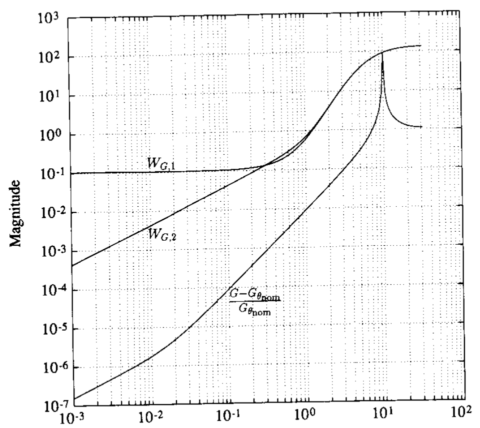

Fig. 3. Frequency response magnitudes of $W_{G,1}$, $W_{G,2}$, and $(G - G_{\theta_{\text{nom}}})/G_{\theta_{\text{nom}}}$.

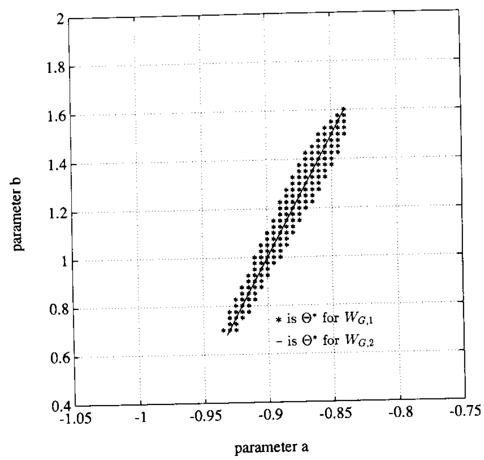

Fig. 4. Limit sets Θ^* for $W_{G,1}$ and $W_{G,2}$.

$T(z)$, then

$$\|T\|_{\mathscr{H}_\infty} = \sup_{|\omega| \le \pi} |T(e^{j\omega})| \tag{33}$$

$$= \sup_{\substack{\|x\|_{k2} \neq 0 \\ k \in \mathbb{N}}} \frac{\|Tx\|_{k2}}{\|x\|_{k2}} \tag{34}$$

$$= \sup\{\gamma: \|Tx\|_{k2} \le \gamma \|x\|_{k2}, \forall \|x\|_{k2} < \infty, \forall k \in \mathbb{N}\}. \tag{35}$$

As a direct consequence, we also have

$$\sup_{\|T\|_{\mathscr{H}_\infty} \le \gamma} \|Tx\|_{k2} = \gamma \|x\|_{k2}. \tag{36}$$

To show that

$$\Theta^* \subseteq \Theta_k, \qquad \forall k \in \mathbb{N} \tag{37}$$

let $\theta^* \in \Theta^*$, i.e.,

$$G = \frac{B_{\theta^*}}{A_{\theta^*}}(1 + \Delta_G^* W_G) \tag{38}$$

with $\|\Delta_G^*\|_{\mathscr{H}_\infty} \le 1$. Note that θ^* and Δ_G^* must agree with the measured data, so

$$A_{\theta^*}y - B_{\theta^*}u = \Delta_G^* W_G B_{\theta^*} u. \tag{39}$$

Taking the l_2-norm, we have

$$\|A_{\theta^*}y - B_{\theta^*}u\|_{k2} = \|\Delta_G^* W_G B_{\theta^*} u\|_{k2}. \tag{40}$$

Since $\|\Delta_G^*\|_{\mathscr{H}_\infty} \le 1$, (36) implies that θ^* must satisfy

$$\|A_{\theta^*}y - B_{\theta^*}u\|_{k2} \le \|W_G B_{\theta^*} u\|_{k2}. \tag{41}$$

Therefore,

$$\theta^* \in \{\theta: \|A_\theta y - B_\theta u\|_{k2} \le \|W_G B_\theta u\|_{k2}\} = \Theta_k \tag{42}$$

for $\Theta^* \subseteq \Theta_k$. From this, it follows immediately that $\Theta^* \subseteq \Theta[N]$. □

A. Frequency-Domain Expressions

Define the asymptotic equation-error set as

$$\Theta_\infty \triangleq \lim_{k \to \infty} \Theta_k. \tag{43}$$

The limit set Θ^* and the asymptotic equation-error set Θ_∞ are expressed in the frequency domain in the following theorem.

Theorem 2:

i) The limit set has the following decomposition:

$$\Theta^* = \Theta^*_{\text{stab}} \cap \Theta^*_{\text{freq}} \tag{44}$$

where

$$\Theta^*_{\text{stab}} = \left\{\theta: \frac{A_\theta G - B_\theta}{W_G B_\theta} \text{ stable}\right\} \tag{45}$$

$$\Theta^*_{\text{freq}} = \{\theta: |A_\theta(e^{j\omega})G(e^{j\omega}) - B_\theta(e^{j\omega})| \le |W_G(e^{j\omega})B_\theta(e^{j\omega})|, \forall |\omega| \le \pi\}. \tag{46}$$

ii) If $y = Gu$ and u has spectrum $S_{uu}(\omega)$, then

$$\Theta_\infty = \left\{\theta: \frac{1}{2\pi}\int_{-\pi}^{\pi} (|A_\theta G - B_\theta|^2 - |W_G B_\theta|^2) S_{uu}\, d\omega \le 0\right\}. \tag{47}$$

Proof: The decomposition of Θ^* follows directly from the definition of the $\mathscr{H}_\infty$ norm. The asymptotic set description is a direct application of the spectral expressions in (9). □

Theorem 1 states that $\Theta^* \subseteq \Theta_k$ for all k. It is clear from the frequency-domain expression for Θ^* that $\Theta^* \subseteq \Theta_\infty$ also because $\theta \in \Theta^*$ implies that the integrand in the frequency-domain expression for Θ_∞ is negative. Note also that the definition of Θ^* describes a parameter set via an $\mathscr{H}_\infty$ norm. By comparison, Θ_∞ is described via an $\mathscr{H}_2$ norm when u is white noise with $S_{uu}(\omega) = 1$, i.e.,

$$\Theta_\infty = \left\{\theta: \frac{\|A_\theta G - B_\theta\|_{\mathscr{H}_2}}{\|W_G B_\theta\|_{\mathscr{H}_2}} \le 1\right\}. \tag{48}$$

B. Use of Data Filtering

The effect of data filtering is to replace (y, u) with (Fy, Fu), where F is a filter with transfer function $F(z)$. Hence

$$\Theta_k = \{\theta: \| A_\theta Fy - B_\theta Fu\|_{k2} \le \| B_\theta W_G Fu\|_{k2}\}. \quad (49)$$

The effect of the filter is seen more clearly in the frequency-domain expression

$$\Theta_\infty = \left\{\theta: \frac{1}{2\pi}\int_{-\pi}^{\pi}(|A_\theta G - B|^2 - |W_G B_\theta|^2)|F|^2 S_{uu}\, d\omega \le 0\right\}. \quad (50)$$

The filter and the input spectrum form the frequency-dependent weight $|F(e^{j\omega})|^2 S_{uu}(\omega)$ which also appears in standard equation-error minimization methods [24].

C. Computing the Equation-Error Set

Ideally, it is desirable to compute $\Theta[N]$. This involves intersecting the N sets

$$\{\Theta_k: k = 1,\cdots, N\}.$$

We start with the following result which presents a convenient form for computing Θ_k.

Theorem 3: Define the following vectors whose elements are sequences:

$$\phi \triangleq \begin{bmatrix}\phi_y \\ \phi_u\end{bmatrix} \quad (51)$$

$$\phi_y \triangleq [-z^{-1}y \cdots -z^{-n}y]^T \quad (52)$$

$$\phi_u \triangleq [z^{-1}u \cdots z^{-m}u]^T. \quad (53)$$

Then,

i) Θ_k can be expressed in the quadratic form

$$\Theta_k = \{\theta: \theta^T\Gamma_k\theta - 2\beta_k^T\theta + \alpha_k \le 0\} \quad (54)$$

where $\alpha_k \in \mathbb{R}$, $\beta_k \in \mathbb{R}^p$, and $\Gamma_k \in \mathbb{R}^{p\times p}$ (with $p = m + n$) are given by

$$\alpha_k = \mathscr{E}_k(y^2) \quad (55)$$

$$\beta_k = \mathscr{E}_k(\phi y) \quad (56)$$

$$\Gamma_k = \mathscr{E}_k(\phi\phi^T) - \begin{bmatrix}0 & 0 \\ 0 & \mathscr{E}_k((W_G\phi_u)(W_G\phi_u)^T)\end{bmatrix}. \quad (57)$$

ii) Provided Γ_k^{-1} exists, another expression is

$$\Theta_k = \left\{\theta: (\theta - \hat{\theta}_k)^T\Gamma_k(\theta - \hat{\theta}_k) \le V_k\right\} \quad (58)$$

$$\hat{\theta}_k = \Gamma_k^{-1}\beta_k \quad (59)$$

$$V_k = \beta_k^T\Gamma_k^{-1}\beta_k - \alpha_k. \quad (60)$$

iii) All the eigenvalues of Γ_k are real and some of them can be negative. When $\Gamma_k > 0$, Θ_k is an ellipsoid in $\mathbb{R}^p$. When Γ_k is indefinite, Θ_k is an hyperboloid in $\mathbb{R}^p$.

Remarks: In part ii), the center of the set $\hat{\theta}_k$ is identical to the ordinary least-squares estimate when $W_G = 0$. This occurs only when nonparametric dynamics are neglected.

Proof: Using the definitions in the theorem, we have

$$A_\theta y - B_\theta u = y - \theta^T\phi \quad (61)$$

$$W_G B_\theta u = \theta^T W_G\begin{bmatrix}0 \\ \phi_u\end{bmatrix}. \quad (62)$$

Hence, substituting into (31), we have

$$\Theta_k = \left\{\theta \in \mathbb{R}^p: \|y - \theta^T\phi\|_{k2} \le \left\|\theta^T W_G\begin{bmatrix}0 \\ \phi_u\end{bmatrix}\right\|_{k2}\right\}. \quad (63)$$

Using (12), the quadratic form of Θ_k follows immediately, which proves part i).

Part ii) is obtained by direct substitution when Γ_k^{-1} exists.

To prove iii), observe that Γ_k can be expressed as follows:

$$\Gamma_k = \begin{bmatrix}\Gamma_{k,11} & \Gamma_{k,12} \\ \Gamma_{k,12}^T & \Gamma_{k,22}\end{bmatrix} \quad (64)$$

where

$$\Gamma_{k,11} = \mathscr{E}_k(\phi_y\phi_y^T) \quad (65)$$

$$\Gamma_{k,12} = \mathscr{E}_k(\phi_y\phi_u^T) \quad (66)$$

$$\Gamma_{k,22} = \mathscr{E}_k(\phi_u\phi_u^T - (W_G\phi_u)(W_G\phi_u)^T). \quad (67)$$

The $\Gamma_{k,22}$ matrix subblock can obviously cause Γ_k to have negative eigenvalues. The square roots of the eigenvalues of Γ_k^{-1} are the lengths of the semiaxes of the ellipsoid. Therefore, as Γ_k becomes singular, some directions of the ellipsoid become unbounded. A hyperboloid results when one or more eigenvalues of Γ_k become negative. □

Note that if the spectrum of u is concentrated at those frequencies where $|W_G(e^{j\omega})|$ is large, the $\Gamma_{k,22}$ matrix subblock can have negative eigenvalues. This tends to make Γ_k become indefinite, so that Θ_k becomes an hyperboloid. This will be illustrated in an example in the next section.

D. Example of Θ_k

The true system was selected, as in the previous example in Section III-B using the weight $W_{G,1}$ defined in (25). The input was a log-spaced sinesweep from 0.1 to 31 rad/s over 102.3 s, thus, $N = 1024$ data samples. Two filtered data sets were generated using eighth-order low-pass Butterworth filters; one with a bandpass of $\omega_f = 2$ rad/s, and the other with $\omega_f = 1$ rad/s.

Fig. 5 shows Θ_{1024} processed with the two data filters. An hyperboloid is obtained with $\omega_f = 2$ rad/s and an ellipsoid with $\omega_f = 1$ rad/s. (Note that only one branch of the hyperboloid is shown in the figure.) This confirms the earlier point that when u is concentrated at those frequencies where $|W_G(e^{j\omega})|$ is large, Θ_k can become unbounded. Points in the limit set Θ^* are shown and, as predicted by the theory, are all contained in the equation-error sets.

E. Computing Intersecting Ellipsoids

To compute $\Theta[N]$ requires computing the intersection of the sets $\{\Theta_k: k = 1,\cdots, N\}$. Since all the Θ_k are convex, it

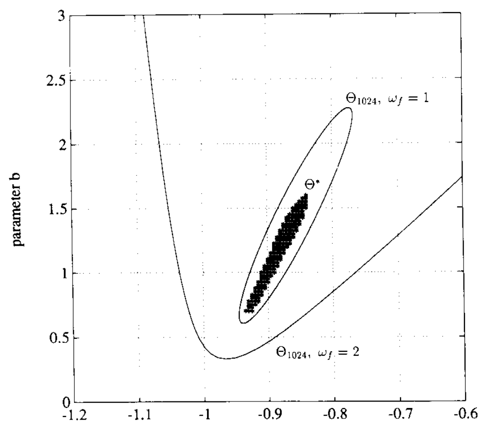

Fig. 5. Θ_{1024} for each data filter; points in limit set are also shown.

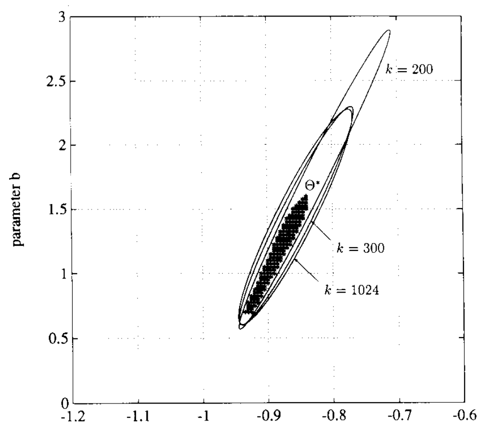

Fig. 6. The equation-error sets $\{\Theta_k: k = 200, 300, \cdots, 1024\}$ using the filtered data with $\omega_f = 1$ rad/s; Θ^* is also shown.

follows that $\Theta[N]$ is convex. In general, it is not, however, an ellipsoid. To see this, we plotted some of the bounding ellipsoid sets in Fig. 6. Specifically, it shows

$$\{\Theta_k: k = 200, 300, \cdots, 1024\}$$

corresponding to the previous example using the data filter with cutoff at 1 rad/s. Observe that the intersection of the sets produces a smaller (convex) set. Several approaches are possible. One approach is to compute the smallest volume ellipsoid that contains the intersection of the ellipsoids. This is discussed in [6] and [3].

F. Effect of Initial Conditions

As defined in Section II, the sequence Gu evaluated at time $t \in \mathbb{N}$ is defined by

$$(Gu)(t) = \sum_{\tau=1}^{t-1} g(\tau) u(t-\tau). \tag{68}$$

To account for initial conditions, let $\tilde{u}$ denote a bounded input applied for $t \le 0$. Thus, the system with initial conditions can be expressed as

$$y = Gu + \tilde{y} \tag{69}$$

with

$$\tilde{y}(t) = \sum_{\tau=t}^{\infty} g(\tau)\tilde{u}(t-\tau), \qquad \forall t \in \mathbb{N}. \tag{70}$$

If G is stable or is in a stabilizing feedback, then $\tilde{y}(t) \to 0$ exponentially as $t \to \infty$. Thus, the effect of initial conditions dies out exponentially fast, or slow, depending on the slowest modes in G or the closed-loop system. Hence, for sufficiently large N, we have $\Theta_N \approx \Theta_\infty$. More precisely, for each $\theta \in \Theta_\infty$,

$$\lim_{N\to\infty} \inf_{\hat{\theta}_N \in \Theta_N} \|\theta - \hat{\theta}_N\| = 0 \tag{71}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^p$. In words, the estimator will eventually report possible parameter values that are close to the asymptotic set, and hence, asymptotically bound the limit set Θ^* as the data length N increases.

Another way to account for the effect of initial condition is to assume bounds on $\tilde{u}$ and the tail of g

$$\sum_{\tau=t}^{\infty} |g(\tau)| \le \kappa_1 \tag{72}$$

$$|\tilde{u}(t)| \le \kappa_2, \qquad t \le 0. \tag{73}$$

Then $|\tilde{y}(t)| \le \kappa_1 \kappa_2$, and it can be treated as a bounded disturbance in (69), see e.g., [30].

G. Other Forms of Nonparametric Uncertainty

The equation-error sets we have developed so far assume a multiplicative form of nonparametric uncertainty. This is not a necessary restriction as they could also have been developed for other forms. The requisite modifications are shown below for some other typical forms.

Theorem 4:

i) *Multiplicative*: If

$$G = \frac{B_\theta}{A_\theta}(1 + \Delta_G W_G), \qquad \|\Delta_G\|_{\mathscr{H}_\infty} \le 1 \tag{74}$$

then

$$\Theta_k = \{\theta: \|A_\theta y - B_\theta u\|_{k2} \le \|W_G B_\theta u\|_{k2}\}. \tag{75}$$

ii) *Additive*: If

$$G = \frac{B_\theta}{A_\theta} + \Delta_G W_G, \qquad \|\Delta_G\|_{\mathscr{H}_\infty} \le 1 \tag{76}$$

then

$$\Theta_k = \{\theta: \|A_\theta y - B_\theta u\|_{k2} \le \|W_G A_\theta u\|_{k2}\}. \tag{77}$$

iii) *Inverse Multiplicative*: If

$$G = \frac{B_\theta}{A_\theta}\left(\frac{1}{1 + \Delta_G W_G}\right), \qquad \|\Delta_G\|_{\mathscr{H}_\infty} \le 1 \tag{78}$$

then

$$\Theta_k = \{\theta: \|A_\theta y - B_\theta u\|_{k2} \le \|W_G A_\theta y\|_{k2}\}. \tag{79}$$

iv) *Feedback*: If

$$G = \frac{B_\theta}{A_\theta}\left(\frac{1}{1 + \Delta_G W_G \dfrac{B_\theta}{A_\theta}}\right), \qquad \|\Delta_G\|_{\mathscr{H}_\infty} \le 1 \tag{80}$$

then

$$\Theta_k = \{\theta: \|A_\theta y - B_\theta u\|_{k2} \le \|W_G B_\theta y\|_{k2}\}. \tag{81}$$

v) *Coprime Factored (Coupled)*: If

$$G = \frac{B_\theta + \Delta_B W_B}{A_\theta + \Delta_A W_A}, \qquad \left\|\begin{bmatrix}\Delta_B \\ \Delta_A\end{bmatrix}\right\|_{\mathscr{H}_\infty} \le 1 \tag{82}$$

then

$$\Theta_k = \left\{\theta: \|A_\theta y - B_\theta u\|_{k2} \le \left\|\begin{bmatrix} W_B u \\ W_A y\end{bmatrix}\right\|_{k2}\right\}. \tag{83}$$

vi) *Coprime Factored (Uncoupled)*: If

$$G = \frac{B_\theta + \Delta_B W_B}{A_\theta + \Delta_A W_A}, \qquad \|\Delta_B\|_{\mathscr{H}_\infty} \le 1, \ \|\Delta_A\|_{\mathscr{H}_\infty} \le 1 \tag{84}$$

then

$$\Theta_k = \{\theta: \|A_\theta y - B_\theta u\|_{k2} \le \|W_B u\|_{k2} + \|W_A y\|_{k2}\}. \tag{85}$$

vii) *All the above set estimates* Θ_k have the property that

$$\Theta^* \subseteq \Theta[N] \subseteq \Theta_k. \tag{86}$$

Proof: The proof of the property $\Theta^* \subseteq \Theta_k$ for all the cases above is similar to the proof for Theorem 1. We will show it for case vi) only. Let $\theta^* \in \Theta^*$, i.e.,

$$\frac{B_{\theta*} + \Delta_B^* W_B}{A_{\theta*} + \Delta_A^* W_A} \tag{87}$$

with

$$\|\Delta_B^*\|_{\mathscr{H}_\infty} \le 1 \text{ and } \|\Delta_A^*\|_{\mathscr{H}_\infty} \le 1. \tag{88}$$

Since θ^*, Δ_A^*, and Δ_B^* must agree with the measured data

$$A_{\theta*} y - B_{\theta*} u = \Delta_B^* W_B u - \Delta_A^* W_A y. \tag{89}$$

Now take the l_2-norm and apply the triangle inequality with (88), θ^* must also satisfy

$$\|A_{\theta*} y - B_{\theta*} u\|_{k2} \le \|W_B u\|_{k2} + \|W_A y\|_{k2}. \tag{90}$$

Therefore,

$$\theta^* \in \{\theta: \|A_\theta y - B_\theta u\|_{k2} \le \|W_B u\|_{k2} + \|W_A y\|_{k2}\} = \Theta_k \tag{91}$$

and $\Theta^* \subseteq \Theta_k$. □

From these forms it is straightforward to generate the corresponding quadratic forms for computing the sets. In those cases, when the right-hand side of any of the above inequalities does not depend on the parameter θ, the center of the parametric set is the usual least-squares estimate, e.g., [32].

V. Equation-Error Set Estimation with Disturbances

There are many ways to characterize the disturbance environment both in terms of the location and the type of disturbance. To simplify the discussion, we assume that the disturbance is located additively at the output, as given by (16)

$$y = Gu + v.$$

The most common type is the stochastic disturbance which we consider in this section. Deterministic "worst-case" types of disturbances are discussed briefly in Section VI.

A. Stochastic Additive Disturbance

Suppose that the disturbance v is a zero-mean quasi-stationary sequence in the set

$$\mathscr{V} = \{v: S_{vv}(\omega) \le \sigma^2 |W_H(e^{j\omega})|^2, \ S_{vu}(\omega) = 0, \forall |\omega| \le \pi\} \tag{92}$$

where $W_H(z)$ is a stable and stably invertible transfer function. Equivalently, we can think of v as the output of a stable uncertain linear-time invariant system H with a white-noise input e. Hence,

$$v = He \tag{93}$$

where H is in the set of linear-time-invariant systems $\mathscr{H}$ and e is in the set of stochastic sequences $\mathscr{W}_{\text{stoch}}$ defined as follows:

$$\mathscr{H} \triangleq \{\Delta_H W_H \text{ stable}: \|\Delta_H\|_{\mathscr{H}_\infty} \le 1\} \tag{94}$$

$$\mathscr{W}_{\text{stoch}} \triangleq \{\text{white noise } e: S_{ee}(\omega) = \sigma_e^2 \le \sigma^2, \ S_{eu}(\omega) = 0, \ \forall |\omega| \le \pi, \ \text{bounded fourth moment}\}. \tag{95}$$

The disturbance set then becomes

$$\mathscr{V} = \{v = He: H \in \mathscr{H}, e \in \mathscr{W}_{\text{stoch}}\}. \tag{96}$$

Assuming that W_H and σ are known, the disturbance set defined above is otherwise parameter-free. One can compare this set description to $\mathscr{G}$ which contains the parametric transfer function $G_\theta(z)$. As it is, the disturbance set is perfectly adequate for describing a sensor noise. However, in the case of a general disturbance reflected to the output, the set merely serves to provide an upper bound. For small disturbances this is adequate, but the set is potentially conservative otherwise. For a more complete discussion on this matter, see [19].

We now have the following.

Theorem 5: Suppose that the true plant which generated $\{y, u: t = 1, \cdots, N\}$ has the structure described above. Then

i)

$$\Theta_N \to \Theta_\infty \qquad \text{w.p. 1 as } N \to \infty. \tag{97}$$

ii)

$$\Theta^* \subseteq \Theta_\infty \tag{98}$$

where the equation-error sets are now defined as follows:

$$\Theta_N \triangleq \Big\{\theta: \mathscr{E}_N\Big(\big[W_H^{-1}(A_\theta y - B_\theta u)\big]^2\Big) \le \mathscr{E}_N\Big(\big(W_H^{-1} W_G B_\theta u\big)^2\Big) + \sigma^2(1 + \theta_A^T \theta_A)\Big\} \quad (99)$$

$$\Theta_\infty \triangleq \Big\{\theta: \bar{\mathscr{E}}\Big(\big[W_H^{-1}(A_\theta y - B_\theta u)\big]^2\Big) \le \bar{\mathscr{E}}\Big(\big(W_H^{-1} W_G B_\theta u\big)^2\Big) + \sigma^2(1 + \theta_A^T \theta_A)\Big\} \quad (100)$$

with

$$\theta_A \triangleq [a_1 \cdots a_n]^T. \quad (101)$$

iii) In the frequency domain

$$\Theta_\infty = \left\{\theta: \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_\theta(\omega)}{|W_H(e^{j\omega})|^2}\, d\omega \le 0\right\} \quad (102)$$

with

$$f_\theta(\omega) = \left(|A_\theta G - B_\theta|^2 - |W_G B_\theta|^2\right) S_{uu}(\omega) + |A_\theta|^2\left(|H|^2 S_{ee}(\omega) - |W_H|^2 \sigma^2\right). \quad (103)$$

Observe that both the finite-data set Θ_N, as well as the infinite-data set Θ_∞, depend on the noise intensity σ and the disturbance weighting transfer function W_H, whose inverse acts as a data filter. The theorem is analogous to the many prediction-error based parameter estimators in the sense that for a sufficiently long data length N, the estimate is equal to the true value with high probability [24]. In our case, the finite-data set Θ_N will contain Θ^* with high probability. Part i) of the theorem means that for each $\theta \in \Theta_\infty$, there is a $\hat{\theta}_N \in \Theta_N$ close to it as N increases. More precisely,

$$\inf_{\hat{\theta}_N \in \Theta_N} \|\theta - \hat{\theta}_N\| \to 0 \qquad \text{w.p. 1 as } N \to \infty \quad (104)$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^p$.

The integrand in the frequency-domain expression for Θ_∞ is always negative provided that for all $|\omega| \le \pi$

$$\left|G - \frac{B_\theta}{A_\theta}\right|^2 - \left|W_G \frac{B_\theta}{A_\theta}\right|^2 \le \frac{1}{S_{uu}}\left(|W_H|^2\sigma^2 - |H|^2 S_{ee}\right). \quad (105)$$

We can now see the usual effects of signal-to-noise ratio. As the noise power σ^2 increases, the "volume" in Θ_∞ will increase. Conversely, if $S_{uu}(\omega)$ is large at many frequencies, Θ_∞ will shrink. In addition, in the frequency ranges where $|W_H(e^{j\omega})| \gg H(e^{j\omega})$, an indication of poor prior information, very large-input power at these frequencies is required to keep Θ_∞ small.

Proof: Under the assumptions, the true system can be expressed as

$$y = \frac{B_\theta}{A_\theta}(1 + \Delta_G W_G)u + \Delta_H W_H e \quad (106)$$

for some $\|\Delta_G\|_{\mathscr{H}_\infty} \le 1$, $\|\Delta_G\|_{\mathscr{H}_\infty} \le 1$, and $e \in \mathscr{W}_{\text{stoch}}$. Rearranging terms and filtering by W_H^{-1} gives

$$W_H^{-1}(A_\theta y - B_\theta u) = \Delta_G W_H^{-1} W_G B_\theta u + \Delta_H A_\theta e. \quad (107)$$

Squaring both sides and taking autocorrelation at $\tau = 0$, we get

$$\bar{\mathscr{E}}\Big(\big[W_H^{-1}(A_\theta y - B_\theta u)\big]^2\Big) = \bar{\mathscr{E}}\Big(\big(\Delta_G W_H^{-1} W_G B_\theta u\big)^2\Big) + \bar{\mathscr{E}}\big((\Delta_H A_\theta e)^2\big) \quad (108)$$

where the cross terms (between e and u) are zero because e and u are independent. Now take the supremum of the right-hand side to obtain the infinite-data parameter set

$$\Theta_\infty = \Big\{\theta: \bar{\mathscr{E}}\Big(\big[W_H^{-1}(A_\theta y - B_\theta u)\big]^2\Big) \le \sup_{\Delta_G, \Delta_H, e}\Big[\bar{\mathscr{E}}\Big(\big(\Delta_G W_H^{-1} W_G B_\theta u\big)^2\Big) + \bar{\mathscr{E}}\big((\Delta_H A_\theta e)^2\big)\Big]\Big\}. \quad (109)$$

To evaluate the right-hand side above, we now use the assumptions $\|\Delta_G\|_{\mathscr{H}_\infty} \le 1$, $\|\Delta_H\|_{\mathscr{H}_\infty} \le 1$, and $e \in \mathscr{W}_{\text{stoch}}$ to obtain

$$\sup_{\|\Delta_G\|_{\mathscr{H}_\infty} \le 1} \bar{\mathscr{E}}\Big(\big(\Delta_G W_H^{-1} W_G B_\theta u\big)^2\Big) = \bar{\mathscr{E}}\Big(\big(W_H^{-1} W_G B_\theta u\big)^2\Big) \quad (110)$$

$$\sup_{e \in \mathscr{W}_{\text{stoch}}} \sup_{\|\Delta_H\|_{\mathscr{H}_\infty} \le 1} \bar{\mathscr{E}}\big((\Delta_H A_\theta e)^2\big) = \sup_{e \in \mathscr{W}_{\text{stoch}}} \bar{\mathscr{E}}\big((A_\theta e)^2\big) \quad (111)$$

$$= \sigma^2 \|A_\theta\|_{\mathscr{H}_2}^2 \quad (112)$$

$$= \sigma^2\left(1 + \sum_{k=1}^{n} a_k^2\right) \quad (113)$$

$$= \sigma^2(1 + \theta_A^T \theta_A). \quad (114)$$

This yields the set Θ_∞ as defined in the theorem.

Observe that Θ_N has precisely the same form as Θ_∞ except that the operator $\bar{\mathscr{E}}(\cdot)$ is replaced everywhere with the sample mean $\mathscr{E}_N(\cdot)$. To show (97), recall from [24, pp. 34–35] that if the stochastic part of x can be described as filtered white noise, then the spectrum of an observed single realization of x, computed as for a deterministic signal, coincides, with probability 1, with that of the process, i.e.,

$$\lim_{N\to\infty} \mathscr{E}_N(x^2) \to \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathscr{E}\big(x(t)^2\big) = \bar{\mathscr{E}}(x^2). \quad (115)$$

The conditions for this convergence are that x is a quasi-stationary sequence and the white noise has bounded fourth moment. Note that since u and y are assumed to have finite power, $W_H^{-1}(A_\theta y - B_\theta u)$ and $W_H^{-1} W_G B_\theta u$ are quasi-stationary. Thus, the convergence in (97) holds.

To show that $\Theta^* \subseteq \Theta_\infty$, we use the frequency-domain expressions in iii). Observe that the frequency-domain expression for Θ_∞ can be obtained by substituting $y = Gu +$

He in (100) to get

$$\Theta_\infty = \left\{\theta: \bar{\mathscr{E}}\left(\left[W_H^{-1}(A_\theta G - B_\theta)u + W_H^{-1}A_\theta He\right]^2\right) \leq \bar{\mathscr{E}}\left(\left(W_H^{-1}W_G B_\theta u\right)^2\right) + \sigma^2\|A_\theta\|_{\mathscr{H}_2}^2\right\}. \quad (116)$$

Now use the fact that u and e are independent to simplify, then apply Parseval's theorem. In the frequency-domain expression, the assumption $H \in \mathscr{H}$ means that

$$|H(e^{j\omega})|^2 S_{ee}(\omega) - |W_H(e^{j\omega})|^2\sigma^2 \leq 0, \qquad \forall\omega \quad (117)$$

and $\theta \in \Theta^*$ means that

$$|A_\theta G - B_\theta|^2 - |W_G B_\theta|^2 \leq 0, \qquad \forall\omega. \quad (118)$$

Thus, $\theta \in \Theta^*$ guarantees that $f_\theta(\omega)$ is negative for all frequencies, and hence, $\Theta^* \subseteq \Theta_\infty$. □

1) Example of Bias Estimation: As an illustrative example, consider estimating a constant in noise

$$y(t) = b_0 + e(t). \quad (119)$$

In this case, $W_G(z) = 0$ to reflect the absence of nonparametric uncertainty, and $H(z) = 1$. In addition, $W_H(z) = 1$, and $H \in \mathscr{H}$ is satisfied. If $e \in \mathscr{W}_{\text{stoch}}$, then the set estimate for b_0 is

$$\Theta_N = \left\{b: (b - \hat{b})^2 \leq \sigma^2 - \mathscr{E}_N\left((y - \hat{b})^2\right)\right\} \quad (120)$$

where $\hat{b} = \mathscr{E}_N(y)$. For large N, the right-hand side behaves as $\sigma^2 - \sigma_0^2$, where σ_0^2 is the true noise variance. Note that the limit set Θ^*, in this case, is the point b_0. Since $\hat{b} \to b_0$ as $N \to \infty$, we see that $\Theta^* \subseteq \Theta_\infty$ as stated in the theorem. Furthermore, as the bounding variance σ approaches σ_0, the set Θ_∞ becomes a point. Observe that Θ_∞ does not shrink to a point when there is nonparametric uncertainty, i.e., $W_G(z) \neq 0$.

B. Computing the Equation-Error Set

For computing Θ_N, we have the following result

Theorem 6: As in Theorem 3, define the vector sequences ϕ, ϕ_y, and ϕ_u. Then:

i) Θ_N can be expressed in the quadratic form

$$\Theta_N = \{\theta: \theta^T\Gamma_N\theta - 2\beta_N^T\theta + \alpha_N \leq 0\} \quad (121)$$

where $\alpha_N \in \mathbb{R}$, $\beta_N \in \mathbb{R}^p$, and $\Gamma_N \in \mathbb{R}^{p\times p}$ are given by

$$\alpha_N = \mathscr{E}_N\left((W_H^{-1}y)^2\right) - \sigma^2 \quad (122)$$

$$\beta_N = \mathscr{E}_N\left((W_H^{-1}\phi)(W_H^{-1}y)\right) \quad (123)$$

$$\Gamma_N = \mathscr{E}_N\left((W_H^{-1}\phi)(W_H^{-1}\phi)^T\right) - \begin{bmatrix} \sigma^2 I_n & 0 \\ 0 & \mathscr{E}_N\left((W_H^{-1}W_G\phi_u)(W_H^{-1}W_G\phi_u)^T\right)\end{bmatrix}. \quad (124)$$

ii) Provided Γ_N^{-1} exists, another expression is

$$\Theta_N = \left\{\theta: (\theta - \hat{\theta}_N)^T\Gamma_N(\theta - \theta_N) \leq V_N\right\} \quad (125)$$

$$\hat{\theta}_N = \Gamma_N^{-1}\beta_N \quad (126)$$

$$V_N = \beta_N^T\Gamma_N^{-1}\beta_N - \alpha_N. \quad (127)$$

iii) When $\Gamma_N > 0$, Θ_N is an ellipsoid in $\mathbb{R}^p$ and when Γ_N is indefinite, Θ_N is a hyperboloid in $\mathbb{R}^p$.

The proof of the above proceeds along the same lines as that of Theorem 3, and is omitted.

The infinite-data parameter set estimate can also be expressed in a form identical to that for the finite-data set

$$\Theta_\infty = \{\theta: \alpha - 2\beta^T\theta + \theta^T\Gamma\theta \leq 0\} \quad (128)$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^p$, and $\Gamma \in \mathbb{R}^{p\times p}$ are given by

$$\alpha = \bar{\mathscr{E}}\left((W_H^{-1}y)^2\right) - \sigma^2 \quad (129)$$

$$\beta = \bar{\mathscr{E}}\left((W_H^{-1}\phi)(W_H^{-1}y)\right) \quad (130)$$

$$\Gamma = \bar{\mathscr{E}}\left((W_H^{-1}\phi)(W_H^{-1}\phi)^T\right) - \begin{bmatrix} \sigma^2 I_n & 0 \\ 0 & \bar{\mathscr{E}}\left((W_H^{-1}W_G\phi_u)(W_H^{-1}W_G\phi_u)^T\right)\end{bmatrix}. \quad (131)$$

C. Example of Θ_N with Disturbance

The example system is as before with G given by (23), and W_G given by (25). The disturbance dynamics is

$$H(z) = \frac{0.1}{z - 0.9} \quad (132)$$

and the disturbance weight is

$$W_H(z) = \frac{1}{\delta_H}H(z) \quad (133)$$

where $\delta_H \in (0, 1)$ is a parameter chosen by the user.

The disturbance v is simulated as the output of H driven by e, a sequence of independently distributed Gaussian variables with zero mean and variance σ^2. Three series of experiments are carried out to study the effects of noise power (choice of σ), mismatch between H and W_H (choice of δ_H), and length of data record (choice of N). In the first two experiments, the input u is a linearly-spaced sinesweep from 0.01 to 0.5 rad/s over 102.3 s, giving $N = 1024$ data samples. In the third experiment, N is varied.

To study the effects of noise power, σ is varied in this experiment. As suggested by Theorem 5, the parameter set estimate should expand as σ increases. This is supported by Fig. 7, where Θ_N is plotted for $\sigma = 0.1, 0.2$, and 0.4. Note that in all cases shown here, $\Theta^* \subseteq \Theta_N$.

In Fig. 8, the value of δ_H is varied from 0.6 to 1.0. Again, as suggested by Theorem 5, as the mismatch between H and W_H becomes larger, i.e., $|\delta_H|$ becomes smaller, Θ_N grows.

The effects of different data record lengths are studied in the last experiment. For the cases of $N = 1024$ and 2048 with $\sigma = 0.5$, and $\delta_H = 1.0$, Θ^* is not in Θ_N. This is still in agreement with our results because in the stochastic disturbance case Θ^* is only guaranted to be in Θ_N as N tends to infinity. As shown in Fig. 9, Θ^* is in Θ_N for $N = 4096$.

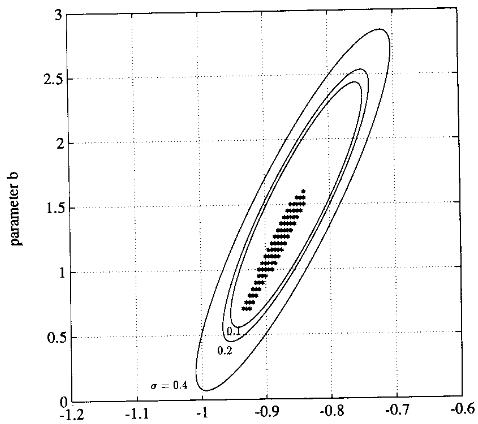

Fig. 7. Θ_N for different values of σ ($N = 1024$, $\delta_H = 0.8$).

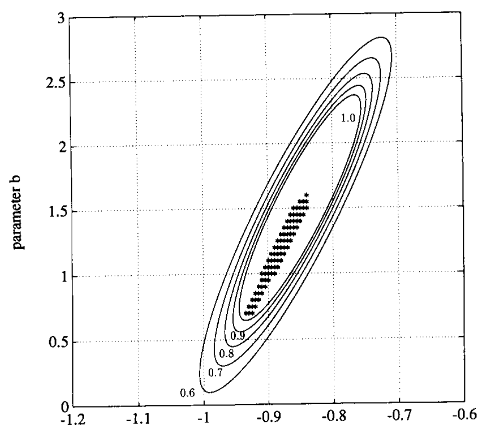

Fig. 8. Θ_N for different values of δ_H ($\sigma = 0.2$, $N = 1024$).

VI. Some Extensions

In this section, we first consider the extension of our results for the equation-error set estimates to the output-error set. We then consider disturbances which are deterministic in nature rather than stochastic, as considered in the previous section.

A. Disturbance-Free Output-Error Set Estimation

The results obtained for the equation-error set in Section IV can be repeated *mutatis mutandis* for the output-error set, but for the notable exception of forming a quadratic set for computational purposes none exists for output-error identification [24].

Theorem 7: Suppose the measured data $\{y, u\colon t = 1, \cdots, N\}$ is generated from $y = Gu$ with $G \in \mathcal{G}$. Then the following holds:

$$\Theta^* \subseteq \Theta^{oe}[N] \subseteq \Theta_k^{oe}, \qquad \forall k \in [1, N], \forall N \in \mathbb{N} \tag{134}$$

where $\Theta^{oe}[N]$ and Θ_k^{oe} are the output-error set estimates

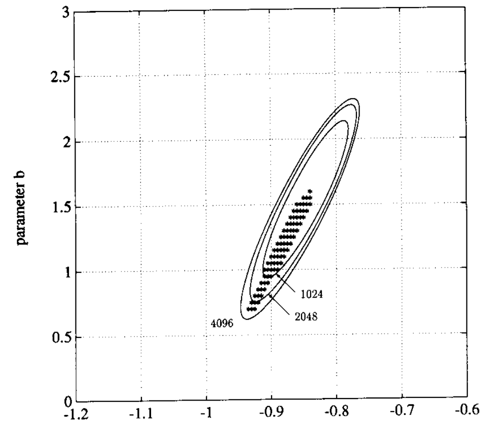

Fig. 9. Θ_N for different values of N ($\sigma = 0.5$, $\delta_H = 1.0$).

given by

$$\Theta_k^{oe} = \left\{\theta\colon \left\| y - \frac{B_\theta}{A_\theta} u \right\|_{k2} \leq \left\| W_G \frac{B_\theta}{A_\theta} u \right\|_{k2} \right\} \tag{135}$$

$$\Theta^{oe}[N] = \bigcap_{k=1}^{N} \Theta_k^{oe}. \tag{136}$$

Remark: We refer to Θ_k^{oe} and $\Theta^{oe}[N]$ as *output-error parameter sets* because the output-error term $y - (B_\theta / A_\theta)u$ appears in their descriptions.

Proof: The proof of $\Theta^* \subseteq \Theta^{oe}[N] \subseteq \Theta_k^{oe}$ is identical to the one for Theorem 1. □

The sets Θ_k and Θ_k^{oe} are both worst-case estimates, both contain Θ^*, but they are not necessarily the same sets for identical input sequences. Another major difference is that both sides of the inequality in Θ_k are affine in θ, whereas in Θ_k^{oe} they are linear fractional in θ. The former property makes it very easy to compute Θ_k, as has been shown, whereas the latter makes it difficult to compute the output-error sets, as usual.

B. Deterministic Additive Disturbances

So far, we have only considered stochastic disturbances. We now briefly examine the effect of deterministic disturbances.

Suppose, as before, that the true system is

$$y = Gu + He \tag{137}$$

with $G \in \mathcal{G}$ and $H \in \mathcal{H}$ as previously described. We now consider the following deterministic set which describes quasi-stationary sequences with bounded spectra:

$$\mathcal{W}_{\text{spec}} = \{e(t)\colon S_{ee}(\omega) \leq \sigma_{\text{spec}}^2, \forall\, |\omega| \leq \pi\}. \tag{138}$$

We then obtain the following.

Theorem 8: If $e \in \mathcal{W}_{\text{spec}}$, then

$$\Theta_k = \left\{\theta\colon \sqrt{\mathcal{E}_k\left(\left[W_H^{-1}(A_\theta y - B_\theta u)\right]^2\right)} \leq \sqrt{\mathcal{E}_k\left(\left(W_H^{-1} W_G B_\theta u\right)^2\right)} + \sigma_{\text{spec}} \sqrt{1 + \theta_A^T \theta_A} \right\} \tag{139}$$

and

$$\Theta^* \subseteq \lim_{k\to\infty} \Theta_k. \tag{140}$$

Proof: The proof of (140) proceeds the same as Theorems 1 and 4. Let $\theta^* \in \Theta^*$, then

$$W_H^{-1}(A_{\theta*}y - B_{\theta*}u) = \Delta_G^* W_H^{-1} W_G B_{\theta*}u + \Delta_H A_{\theta*}e. \tag{141}$$

After squaring both sides and taking the sample averages, the Schwarz's inequality, $\|\Delta_G^*\|_{\mathscr{H}_\infty} \le 1$, and $\|\Delta_H\|_{\mathscr{H}_\infty} \le 1$ are applied to obtain

$$\sqrt{\mathscr{E}_k\left(\left[W_H^{-1}(A_{\theta*}y - B_{\theta*}u)\right]^2\right)} \le \sqrt{\mathscr{E}_k\left(\left(W_H^{-1}W_G B_{\theta*}u\right)^2\right)} + \sqrt{\mathscr{E}_k\left(\left(A_{\theta*}e\right)^2\right)}. \tag{142}$$

Now let $k \to \infty$, we have

$$\theta^* \in \left\{\theta: \sqrt{\bar{\mathscr{E}}\left(\left[W_H^{-1}(A_\theta y - B_\theta u)\right]^2\right)} \le \sqrt{\bar{\mathscr{E}}\left(\left(W_H^{-1}W_G B_\theta u\right)^2\right)} + \sigma_{\text{spec}}\sqrt{1+\theta_A^T\theta_A}\right\} \tag{143}$$

and $\Theta^* \subseteq \lim_{k\to\infty}\Theta_k$. □

In both cases of stochastic and deterministic disturbances, the limit set Θ^* is contained in the set estimate as the data length tends to infinity. However, in the deterministic case here, the probabilistic convergence need not be considered. The reason that both cases can be handled in the same way is because a common framework is used for deterministic and stochastic signals, [see (5) and (6)]. Note that instead of using $\mathscr{W}_{\text{spec}}$ to describe the deterministic disturbance e, we can also use

$$\mathscr{W}_{\text{rms}} = \left\{ e(t): \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} e(t)^2 \le \sigma_{\text{rms}}^2 \right\} \tag{144}$$

to describe e and obtain results similar to Theorem 8.

VII. Concluding Remarks

The set-membership approach to system identification starts with the assumption that the underlying true system which generated the measured data is in a known set characterized by some unknown parameters and unknown but bounded nonparametric dynamics. We then derived set estimates for these unknown parameters. In the disturbance-free case, the set estimate has the property that it always contains the limit set. In the presence of stochastic disturbances, the set estimate is shown to have the property that it contains the limit set with probability one as the data length tends to infinity.

The set estimates derived in this paper also have some nice properties for computation. For the equation-error estimates, the set expressions are quadratic in the parameters. Thus, the set estimates are either ellipsoids or hyperboloids in the parameter space. Furthermore, these sets are easily obtained by computing averages of the filtered input-output data. However, when the output-error form is used in the set estimate, these nice properties are lost, which is typical with output-error identification.

The next step is to use these set estimates with a robust on-line control design procedure. One approach would be to bury the parameter uncertainty in another nonparametric uncertainty by finding an overbounding frequency-dependent weighting function. This is a potentially very conservative approach. Alternatively, the minimax approach in [22] and [21] presents a robust control-design procedure to handle the specific type of parameter uncertainty as represented by the ellipsoidal sets. This is a current topic of our research.

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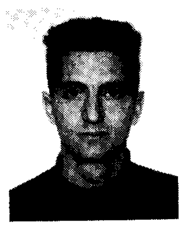

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