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Robust Optimal Control of Linear Discrete-Time Systems using Primal-Dual Interior-Point Methods

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Abstract

In this paper is described how to efficiently solve a robust optimal control problem using recently developed primal-dual interior-point methods. Among potential applications are model predictive control. The optimization problem considered consists of a worst case quadratic performance criterion over a finite set of linear discrete-time models subject to inequality constraints on the states and control signals. The scheme has been prototyped in Matlab. To give a rough idea of the efficiencies obtained, it is possible to solve problems with more than 1000 variables and 5000 constraints in a few minutes on a workstation.

1 Introduction

In recent years rapid progress has been made in solving convex optimization problems. Especially the development of interior-point methods have contributed to this. They have their roots in Karmarkar's method to solve linear programs, [Kar84], and they were extended to nonlinear convex problems by Nesterov and Nemirovski, [NN94].

One particular class of these convex problems are linear matrix inequalities, which have seen many applications in control theory, e.g. [BEGFB94]. Some other convex optimization problems in control are described in [BB91].

Recently specially tailored interior-point methods for model predictive control have appeared. These algorithms solve the resulting quadratic program by utilizing the special structure of the control problem. By ordering the equations and variables in a certain way the linear system of equations that has to be solved for the search directions becomes block-diagonal, [Wri93, Wri96]. By further examining this structure it is possible to diagonalize the matrix using a Riccati-recursion. This makes the computational burden to grow only linearly with the time horizon, [RWR97]. The scope of this paper is to extend these results to robust model predictive control. The robustness is obtained by considering worst case performance over a finite set of models. Numerical algorithms for robust model predictive control has also been proposed in [Bad96].

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The remaining part of the paper is organized as follows. In Section 2 the robust optimal control problem is described. In Section 3 the primal-dual interior-point method is introduced, and it is shown how it can be made to work in an efficient way. Then in Section 4 the method is evaluated on a double-tank example. Finally, in Section 5 some concluding remarks are given.

2 Control Problem

In this section the control problem is described. First the dynamic model is described together with the state and control signal constraints. Then the performance criterion is introduced. Finally is discussed how the problem can be rewritten as a robust quadratic program.

Consider the following linear and time-varying models for $i = 1, \ldots, L$:

$$x_i(k+1) = A_i(k)x_i(k) + B_i(k)u(k), \quad k = 0, \ldots, N-1 \tag{1}$$

$$d_i(k) \geq C_i(k)x_i(k) + D_i(k)u(k), \quad k = 0, \ldots, N \tag{2}$$

where $x_i(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control signal, and where $A_i(k) \in \mathbb{R}^{n\times n}$, $B_i(k) \in \mathbb{R}^{n\times m}$, $C_i(k) \in \mathbb{R}^{p\times n}$, $D_i(k) \in \mathbb{R}^{p\times m}$, and $d_i(k) \in \mathbb{R}^p$. The initial state $x_i(0)$ is given and could be different for each model. The inequality in (2) should be interpreted as component-wise inequality. Notice that this model is rich enough to describe most objectives encountered in model predictive control, see [ML97]. In Section 3 it will be seen how state constraints, control signal constraints and slew rate constraints on the control signal can be incorporated. Since the model is time-varying also non-linear models can be considered as in [Wri93]. This will not be pursued further in this paper.

Introduce the following performance criteria:

$$\phi_i = \frac{1}{2}\sum_{k=0}^{N} \begin{bmatrix} x_i(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q_i(k) & S_i(k) \\ S_i^T(k) & R_i(k) \end{bmatrix} \begin{bmatrix} x_i(k) \\ u(k) \end{bmatrix}, \; i = 1, \ldots, L \tag{3}$$

They could easily be extended to contain linear terms, but this is not done for clarity of exposition. The following optimization problem is the robust optimal control problem that will be considered in this paper:

$$\begin{aligned} &\text{minimize } \gamma \\ &\text{subject to } \phi_i \leq \gamma, \qquad i = 1, \ldots, L \end{aligned}$$

and (1–3). This can with suitable definitions of matrices and variables be written as the robust quadratic program:

$$\text{minimize } \gamma \tag{4}$$

$$\text{subject to } x^T Q_i x \leq \gamma, \quad i = 1, \ldots, L \tag{5}$$

$$Fx = g; \quad Cx \leq d \tag{6}$$

where $x \in \mathbb{R}^{(N+1)(Ln+m)}$, $g \in \mathbb{R}^{(N+1)Ln}$, $d \in \mathbb{R}^{(N+1)Lp}$. How to do this is shown in Appendix A.

3 Interior-Point Method

In this section the Karush-Kuhn-Tucker (KKT) conditions for the robust quadratic program are presented together with an interior point-method that finds a solution for them. Special attention is given to how to compute the search direction for the Newton steps in an efficient way.

The KKT conditions for the robust quadratic program in (4–6) are, see [Wri97]:

$$\sum_{i=1}^{L} Q_i \mu_i x + F^T \pi + C^T \lambda = 0; \quad \sum_{i=1}^{L} \mu_i = 1$$

$$\frac{1}{2} x^T Q_i^T x + s_i = \gamma, \quad i = 1, \ldots, L$$

$$Cx + t = d; \quad Fx = g$$

$$\mu_i s_i = 0; \quad \lambda_i t_i = 0$$

and $(\mu, \lambda, s, t) \geq 0$, where γ and x are the primal variables, $\pi \in \mathbb{R}^{(N+1)Ln}$ is the dual variable associated with the equality constraint, $\mu \in \mathbb{R}^L$ and $\lambda \in \mathbb{R}^{(N+1)Lp}$ are the dual variables associated with the inequality constraints, and where $s \in \mathbb{R}^L$ and $t \in \mathbb{R}^{(N+1)Lp}$ are the slack variables. Define

$$Q = \begin{bmatrix} Q_1 & \cdots & Q_L \end{bmatrix}; \quad X = \text{block diag}_{i=1,\ldots,L}(x)$$

$$Q_\mu = \sum_{i=1}^{L} Q_i \mu_i; \quad S = \text{diag}_{i=1,\ldots,L}(s_i)$$

$$M = \text{diag}_{i=1,\ldots,L}(\mu_i); \quad L = \text{diag}_{i=1,\ldots,(N+1)Lp}(\lambda_i)$$

Introduce $\mathcal{F}(z)$ as

$$\begin{bmatrix} 0 & -\mathbf{1}^T & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{1} & 0 & I & \frac{1}{2}X^T Q^T & 0 & 0 & 0 \\ 0 & 0 & M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_\mu & F^T & C^T & 0 \\ 0 & 0 & 0 & F & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \gamma \\ \mu \\ s \\ x \\ \pi \\ \lambda \\ t \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ g \\ d \\ 0 \end{bmatrix}$$

where $z = (\gamma, \mu, s, x, \pi, \lambda, t)$. Then the KKT conditions can be written as $\mathcal{F}(z) = 0$ and $(\mu, \lambda, s, t) \geq 0$. Primal-dual interior-point methods generate iterates z^j, $j = 1, 2, \ldots$, with $(\mu^j, \lambda^j, s^j, t^j) > 0$ that approach the solution of the KKT conditions as $j \to \infty$. The search directions are Newton-like directions for the equality conditions. Dropping the iteration index j and denoting the current iterate by z, the general linear system to be solved for the search direction can be written

$$\begin{bmatrix} 0 & -\mathbf{1}^T & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{1} & 0 & I & X^T Q^T & 0 & 0 & 0 \\ 0 & S & M & 0 & 0 & 0 & 0 \\ 0 & QX & 0 & Q_\mu & F^T & C^T & 0 \\ 0 & 0 & 0 & F & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & T & L \end{bmatrix} \begin{bmatrix} \Delta\gamma \\ \Delta\mu \\ \Delta s \\ \Delta x \\ \Delta\pi \\ \Delta\lambda \\ \Delta t \end{bmatrix} = r \tag{7}$$

where $T = \text{diag}_{i=1,\ldots,(N+1)Lp}(t_i)$. Notice that the matrix of the linear system of equations is the Jacobian of the nonlinear equations specifying the KKT conditions. Different primal-dual methods are obtained depending on what right-hand side vector r is used. The method used in this paper is a so-called predictor-corrector infeasible-interior-point method and it is described in [Wri97, p. 166]. This algorithm typically converges to an accuracy where the 2-norm of the constraints are smaller than 10^{-4} in about 20 steps irrespective of the size of the problem. The largest computational burden is in computing the search direction. The rest of this section is devoted to how this can be done in an efficient way. Notice that

$$\Delta t = -C\Delta x - Cx - t + d$$

$$\Delta\lambda = -T^{-1}(L(t + \Delta t) - \sigma\nu\mathbf{1})$$

from (7) By substituting this back into (7) the following equation is obtained:

$$\begin{bmatrix} 0 & -\mathbf{1}^T & 0 & 0 & 0 \\ -\mathbf{1} & 0 & I & X^T Q^T & 0 \\ 0 & S & M & 0 & 0 \\ 0 & QX & 0 & Q_\mu + C^T T^{-1} LC & F^T \\ 0 & 0 & 0 & F & 0 \end{bmatrix} \begin{bmatrix} \Delta\gamma \\ \Delta\mu \\ \Delta s \\ \Delta x \\ \Delta\pi \end{bmatrix} = \begin{bmatrix} r_\gamma \\ r_\mu \\ r_s \\ r_x \\ r_\pi \end{bmatrix}$$

where explicit expressions for the right hand side is omitted due to space limitations. The bottom right 2×2 block matrix is much larger than the rest of the matrix, so it makes sense to partition the problem in the following way:

$$\begin{bmatrix} T_1 & T_{12} \\ T_{12}^T & T_2 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

where the definition of the matrices and vectors is obvious from what was said above. Making use of the special structure of T_{12} some calculations show that Δ_1 and Δ_2 can be obtained by solving

$$T_2 \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} = \begin{bmatrix} QX & r_x \\ 0 & r_\pi \end{bmatrix} \tag{8}$$

$$\left(T_1 - \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^T Q^T \xi_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \Delta_1 = U_1 - \begin{bmatrix} 0 \\ X^T Q^T \xi_{12} \\ 0 \end{bmatrix} \tag{9}$$

$$\Delta_2 = \begin{bmatrix} \xi_{12} \\ \xi_{22} \end{bmatrix} - \begin{bmatrix} \xi_{11} \\ \xi_{21} \end{bmatrix} \Delta\mu \tag{10}$$

The dimension of the first set of equations is $(N+1)(2Ln+m)$, whereas the second set of equations has dimension $2L+1$. Hence the second set can be solved with any standard solver. The first set can be efficiently solved using a Riccati recursion in a similar way as in [RWR97]. This is described in more detail in Appendix B. It is there shown that the Riccati recursion for the robust case is L^2 times bigger than for the non-robust case. However, the computational complexity still grows only linearly with N.

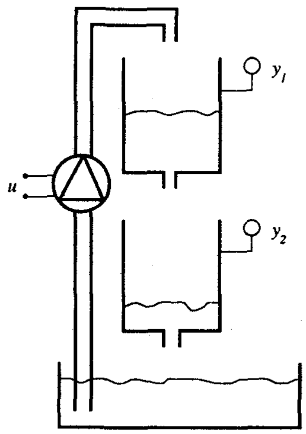

Figure 1: The double-tank process.

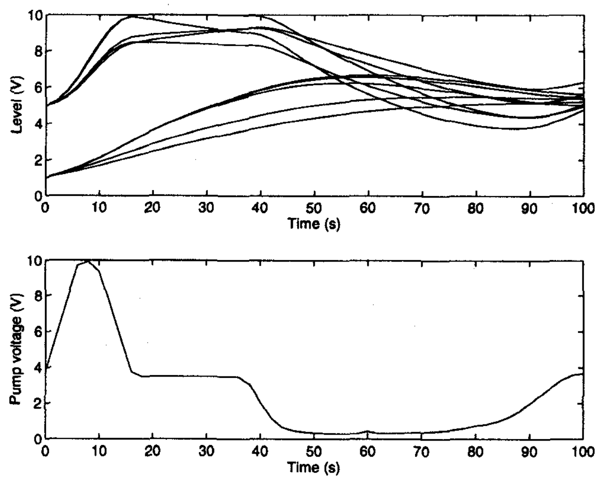

Figure 2: Plot of the performance of the robust optimal control signal.

4 Example

In this section the optimization scheme presented in the previous sections is evaluated on a a double-tank process. Comparisons are made between the robust optimal solution and a non-robust nominal optimal solution.

The double-tank process has been described in [ÅÖ86]. It is depicted in Figure 1. For short reference the model will be given below. The process can be described with

$$A_1\frac{dh_1}{dt} = q - a_1\sqrt{2gh_1}$$
$$A_2\frac{dh_2}{dt} = a_1\sqrt{2gh_1} - a_2\sqrt{2gh_2}$$

where h_1 and h_2 are the levels in the upper and lower tank respectively, q is the water flow into the upper tank, $A_1 = A_2 = 2734 \times 10^{-6}\text{m}^2$ are the areas of the cross-sections of the tanks, and $a_1 = a_2 = 7 \times 10^{-6}\text{m}^2$ are the areas of the cross-sections of the outlet pipes. The height of the tanks are 0.2 m, and the level of the tanks are measured with sensors that give an output voltage y_i proportional to the level. The proportional constant is 50 V/m. Thus the sensor output is limited within the range $[0, 10]$ V. Further the flow q is given by $q = ku$, where u is the pump voltage-input and where $k = 27 \times 10^{-7}\text{m}^3/\text{Vs}$. The voltage u is limited such that $u \in [0, 10]$ V. Linearizing around the steady state solution $y_i^0 = 5$V and $u_0 = 1.82$V yields

$$\frac{d\Delta y}{dt} = \begin{bmatrix} -\alpha_1 & 0 \\ \alpha_{12} & -\alpha_2 \end{bmatrix}\Delta y + \begin{bmatrix} \beta \\ 0 \end{bmatrix}\Delta u$$

where $\alpha_1 = \alpha_{12} = \alpha_2 = 0.0179\text{s}^{-1}$ and $\beta = 0.0494\text{s}^{-1}$. Notice that $\Delta u \in [-1.82, 8.19]$V and that $\Delta y_i \in [-5, 5]$V. The time constants of the tanks are approximately 60s. Hence a reasonable sample interval is $T = 2$s. Sampling with zero order hold yields the discrete time equation

$$\Delta y(k+1) = \Phi\Delta y(k) + \Gamma\Delta u(k)$$

where

$$\Phi = \begin{bmatrix} 0.9648 & 0 \\ 0.0345 & 0.9648 \end{bmatrix}; \quad \Gamma = \begin{bmatrix} 0.0971 \\ 0.0017 \end{bmatrix}$$

To further demonstrate the flexibility of the optimization problem considered a slew-rate constraint will be imposed on u, i.e. the absolute value of du/dt will be limited to 1V/s. This can be accommodated by augmenting the state:

$$\begin{bmatrix} \Delta y(k+1) \\ \Delta\xi(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Delta y(k) \\ \Delta\xi(k) \end{bmatrix} + \begin{bmatrix} \Gamma \\ 1 \end{bmatrix}\Delta u(k)$$

How the matrices A_i, B_i, C_i, and D_i, and the vector d_i of the robust optimal control problem are defined in more detail is omitted because of space limitations. Here it is only noted that different matrices are obtained for different indexes i by taking α_1, α_{12}, α_2, and β equal to their nominal values plus a random normally distributed term with relative standard deviation of 0.25. The performance indexes considered are

$$\phi_i = \sum_{k=0}^{N} \Delta y_2^2(k), \quad i = 1, 2, \ldots, L$$

The result for $L = 5$ and $N = 50$ and an initial value of $y(0) = [5 \quad 1]^T$V is shown in Figure 2. Notice how the different constraints become active. First the slew rate constraint on the control signal is active. Then the control signal reaches its maximal value at 8s. After another 8s the maximum level is reached in the upper tank for one of the models. This is balanced with a constant control signal of 4V for about 20 seconds. Then the control signal is lowered to its minimal value so that the level of the upper tank can decrease and approach the same level as the lower tank level at the final time of 100s.

To get a feel for how different a robust optimal design is as compared to a nominal design, i.e. the design obtained when $L = 1$, both robust and nominal optimal solutions were computed for a range of values of the upper and lower saturation levels of the upper tank. More precisely the constraints were modified to $\Delta y_1 \in [-\Delta y_{\max}, \Delta y_{\max}]$, where $\Delta y_{\max} = \{0.5, 1, 1.5, \ldots, 6\}$. The result of this is presented in Figure 3, where the left plot shows the robust and nominal solutions evaluated on only the nominal model, and where the right plot shows the the robust and nominal solutions

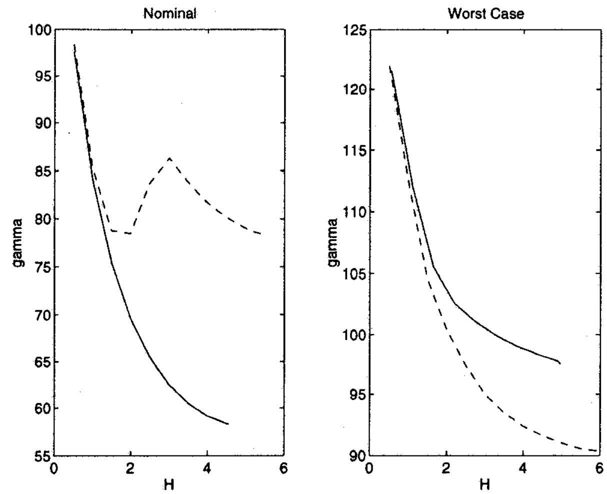

Figure 3: Tradeoff curve between performance index γ and the largest absolute value H of the deviation of the level in the upper tank from half level.

evaluated on the worst case model. It is seen that the robust solution is worse than the nominal solution when evaluated on the nominal model. However, the robust solution outperforms the nominal solution when evaluated on the worst case model.

This section is concluded with the comment that there is not yet any good theoretical explanation of how good the primal-dual interior-point method used works in general. For example it was found that scaling the variables by a factor of 0.2 made a big impact on the speed of the solver.

5 Conclusions

In this paper it has been shown how in an efficient way a primal-dual interior-point method can be used to solve robust optimal control with potential applications to model predictive control. It has been shown that the computational complexity grows linearly with the time horizon N just as for the non-robust case. However, the Riccati recursion is L^2 times larger for the robust case. Still it is possible to solve large problems in a reasonable time on a work station. It is believed that more efficient implementations can be made. Also it is believed that the size of the Riccati recursion can be brought down to the same size as for the non-robust case if an approximate Riccati recursion is used as a pre-conditioner for a conjugate gradient method. This is a topic for further research.

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6 Appendix A

In this appendix it shown how the robust optimal control problem can be rewritten as a robust quadratic program. To this end introduce the extended state and system matrices

$$A(k) = \text{block diag}_{i=1,\dots,L}(A_i(k))$$
$$C(k) = \text{block diag}_{i=1,\dots,L}(C_i(k))$$
$$B(k) = \begin{bmatrix} B_1(k) \\ \vdots \\ B_L(k) \end{bmatrix}; \quad D(k) = \begin{bmatrix} D_1(k) \\ \vdots \\ D_L(k) \end{bmatrix}$$
$$x(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_L(k) \end{bmatrix}; \quad d(k) = \begin{bmatrix} d_1(k) \\ \vdots \\ d_L(k) \end{bmatrix}$$

Then it holds that

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad k = 0,\dots,N-1$$
$$d(k) \geq C(k)x(k) + D(k)u(k), \quad k = 0,\dots,N$$

Define

$$x = \begin{bmatrix} x(0) \\ u(0) \\ \vdots \\ x(N) \\ u(N) \end{bmatrix}; \quad \bar{Q}_i(k) = \begin{bmatrix} Q_i(k) & S_i(k) \\ S_i^T(k) & R_i(k) \end{bmatrix}$$

$$E_i = \begin{bmatrix} 0 & \cdots & 0 & I_n & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_m \end{bmatrix}$$

where the identity matrix in the first block row is in the ith block column, and where the identity matrix in the last block row is in the $(L+1)$th block column, Then it holds that

$$\phi_i = \frac{1}{2}x^T Q_i x, \quad Q_i = \text{block diag}_{k=0,\dots,N}\left(E_i^T \bar{Q}_i(k) E_i\right)$$

Let

$$F = \begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 \\ -A(0) & -B(0) & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -A(N-1) & -B(N-1) & I & 0 \end{bmatrix}$$

$$g = \begin{bmatrix} x^T(0) & 0 & \cdots & 0 \end{bmatrix}^T$$

$$C = \begin{bmatrix} C(0) & D(0) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & C(N) & D(N) \end{bmatrix}; \quad d = \begin{bmatrix} d(0) \\ d(1) \\ \vdots \\ d(N) \end{bmatrix}$$

Then

$$Fx = g; \quad Cx \leq d$$

and hence it has been shown how to rewrite the robust optimal control problem as the robust quadratic program in (4-6).

7 Appendix B

In this appendix it will be shown how the search direction for the Newton step can be efficiently computed using a Riccati recursion. Write (8) as

$$\begin{bmatrix} Q_\mu + C^T T^{-1} L C & F^T \\ F & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{\pi} \end{bmatrix} = \begin{bmatrix} V_x \\ V_\pi \end{bmatrix}$$

where

$$\bar{x} = \begin{bmatrix} \bar{x}(0) \\ \bar{u}(0) \\ \vdots \\ \bar{x}(N) \\ \bar{u}(N) \end{bmatrix}; \; \bar{\pi} = \begin{bmatrix} \bar{\pi}(0) \\ \vdots \\ \bar{\pi}(N) \end{bmatrix}; \; V_x = \begin{bmatrix} \bar{r}_x(0) \\ \bar{r}_u(0) \\ \vdots \\ \bar{r}_x(N) \\ \bar{r}_u(N) \end{bmatrix}; \; V_\pi = \begin{bmatrix} \bar{r}_\pi(0) \\ \vdots \\ \bar{r}_\pi(N) \end{bmatrix}$$

Reorder the equations and variables to get:

$$\begin{bmatrix} 0 & I & 0 & 0 & \cdots & 0 \\ I & P_1(0) & P_{12}(0) & -A^T(0) & \cdots & 0 \\ 0 & P_{12}^T(0) & P_2(0) & -B^T(0) & \cdots & 0 \\ 0 & -A(0) & -B(0) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & P_1^T(N) & P_{12}(N) \\ 0 & \cdots & 0 & 0 & P_{12}^T(N) & P_2(N) \end{bmatrix} \begin{bmatrix} \bar{\pi}(0) \\ \bar{x}(0) \\ \bar{u}(0) \\ \vdots \\ \bar{\pi}(N) \\ \bar{x}(N) \\ \bar{u}(N) \end{bmatrix} = \begin{bmatrix} \bar{r}_\pi(0) \\ \bar{r}_x(0) \\ \bar{r}_u(0) \\ \vdots \\ \bar{r}_\pi(N) \\ \bar{r}_x(N) \\ \bar{r}_u(N) \end{bmatrix}$$

where

$$\begin{bmatrix} P_1(k) & P_{12}(k) \\ P_{12}^T(k) & P_2(k) \end{bmatrix} = \sum_{i=1}^{L} E_i^T \bar{Q}_i(k) E_i \mu(i) + \begin{bmatrix} C(k) & D(k) \end{bmatrix}^T \Sigma(k) \begin{bmatrix} C(k) & D(k) \end{bmatrix}$$

and where $\Sigma(k)$ is defined via

$$\text{block diag}_{k=0,\dots,N}(\Sigma(k)) = LT^{-1}$$

It can be shown using induction that there exist sequences of matrices $\Pi(k) \in \mathbb{R}^{Ln \times Ln}$, and $\Psi(k) \in \mathbb{R}^{Ln \times (L+1)}$ such that

$$\bar{\pi}(k) + \Pi(k)\bar{x}(k) = \Psi(k)$$

These can be recursively computed from

$$\begin{aligned} \Pi(N) &= P_1(N) - P_{12}(N)P_2^{-1}(N)P_{12}^T(N) \\ \Psi(N) &= \bar{r}_x(N) - P_{12}(N)P_2^{-1}(N)\bar{r}_u(N) \\ G(k-1) &= P_2(k-1) + B^T(k-1)\Pi(k)B(k-1) \\ \Pi(k-1) &= P_1(k-1) + A^T(k-1)\Pi(k)A(k-1) \\ &\quad - \left(P_{12}(k-1) + A^T(k-1)\Pi(k)B(k-1)\right) \\ &\quad \times G^{-1}(k-1) \\ &\quad \times \left(P_{12}(k-1) + A^T(k-1)\Pi(k)B(k-1)\right)^T \\ \Psi(k-1) &= \bar{r}_x(k-1) \\ &\quad - \left(P_{12}(k-1) + A^T(k-1)\Pi(k)B(k-1)\right) \\ &\quad \times G^{-1}(k-1)\Big[\bar{r}_u(k-1) \\ &\quad - B^T(k-1)\left(\Pi(k)\bar{r}_\pi(k-1) - \Psi(k)\right)\Big] \\ &\quad - A^T(k-1)\left(\Pi(k)\bar{r}_\pi(k-1) - \Psi(k)\right) \end{aligned}$$

where $\Pi(k)$ obeys a backward Riccati recursion. The solution to the linear set of equations can then be obtained from the forward recursion

$$\begin{aligned} \bar{x}(0) &= \bar{r}_\pi(0) \\ \bar{u}(k-1) &= G^{-1}(k-1)\Big[\bar{r}_u(k-1) - [P_{12}(k-1) \\ &\quad + A^T(k-1)\Pi(k)B(k-1)]\bar{x}(k-1) \\ &\quad - B^T(k-1)\left(\Pi(k)\bar{r}_\pi(k-1) - \Psi(k)\right)\Big] \\ \bar{x}(k) &= A(k-1)\bar{x}(k-1) + B(k-1)\bar{u}(k-1) + \bar{r}_\pi(k-1) \\ \bar{\pi}(k) &= -\Pi(k)\bar{x}(k) + \Psi(k) \end{aligned}$$

Notice that the dimension of the matrix $\Pi(k)$ is L^2 times bigger for the robust case as compared to the non-robust case.