Robust Chebyshev FIR Equalization

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Abstract—**In Chebyshev finite-impulse response (FIR) equalization, we design an FIR filter that minimizes the Chebyshev equalization error, *i.e.*, the maximum absolute deviation between the equalized and the desired frequency response functions, assuming the unequalized response function is known exactly. In robust Chebyshev FIR equalization, we take into account uncertainty in the unequalized response function, described as a set of possible values for the unequalized response at each frequency, by designing an FIR filter that minimizes worst-case Chebyshev equalization error over all possible unequalized response functions. When the uncertainty in unequalized response function is described by a complex uncertainty ellipsoid, at each frequency, we show that the robust Chebyshev FIR equalization design problem can be formulated as a semidefinite program (SDP), and therefore efficiently (and globally) solved. When the uncertainty is given by a complex disk, the design problem can be formulated as a second-order cone program (SOCP), which can be solved almost as fast as the nominal Chebyshev equalization problem (ignoring uncertainty). The robust equalizer design method is demonstrated with a numerical example.**

I. Introduction

A. Chebyshev FIR equalization

Let $G(\omega) \in \mathbf{C}$, defined over $[0, 2\pi]$, be an unequalized frequency response, where ω is the discrete-time frequency variable. In an *equalizer design* problem, we are given a desired frequency response $G^{\rm des}(\omega)$, and want to design an FIR (finite impulse response) equalizer filter with frequency response $H(\omega)$ so that the product $H(\omega)G(\omega)$ approximates $G^{\rm des}(\omega)$ over $[0, 2\pi]$ as well as possible:

$$H(\omega)G(\omega) \approx G^{\rm des}(\omega) \quad \text{for all} \;\; \omega \in [0, 2\pi].$$

(A common choice for $G^{\rm des}(\omega)$ is a delay, $G^{\rm des}(\omega) = e^{-iD\omega}$; in this case, equalization is approximate deconvolution up to delay D.) In practice we often want to achieve this approximation at the frequencies $\omega_1, \ldots, \omega_M$ sampled over $[0, 2\pi]$:

$$H_m G_m \approx G_m^{\rm des}, \qquad m = 1, \ldots, M, \tag{1}$$

where we use the shorthand notation

$$H_m = H(\omega_m), \qquad G_m = G(\omega_m), \qquad G_m^{\rm des} = G^{\rm des}(\omega_m).$$

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We refer to m as the frequency index. We judge the performance of the equalizer by the maximum absolute deviation over the sampled frequencies [1]:

$$C(h) = \max_m \left| G_m H_m - G_m^{\rm des} \right|.$$

In this paper, we focus on constructing equalizers using FIR filters with real coefficients, and therefore assume that the unequalized frequency responses are Hermitian; our results are readily extended to handle the case where the FIR coefficients are complex. We use $h = (h_0, h_1, \ldots, h_{n-1}) \in \mathbf{R}^n$ to denote the impulse response of the FIR equalizer filter $H : [0, \pi] \to \mathbf{C}$, where

$$H(\omega) = \sum_{k=0}^{n-1} h_k e^{-ik\omega}.$$

Note that $H(\omega)$ is linear in the coefficients h:

$$H(\omega) = w^* h,$$

where $(\cdot)^*$ denotes Hermitian transpose and

$$w = (1, e^{i\omega}, \ldots, e^{i(n-1)\omega}) \in \mathbf{C}^n.$$

The maximum absolute deviation can now be expressed as

$$C(h) = \max_m \left| G_m w_m^* h - G_m^{\rm des} \right|,$$

where $w_m = (1, e^{i\omega_m}, \ldots, e^{i(n-1)\omega_m}) \in \mathbf{C}^n$.

The *Chebyshev FIR equalizer design* problem is to find filter coefficients $h \in \mathbf{R}^n$ that minimize the maximum absolute deviation:

$$\text{minimize} \quad C(h), \tag{2}$$

where the problem data are the complex numbers G_m, $G_m^{\rm des}$, $m = 1, \ldots, M$. Any solution of (2) is called a *nominal optimal equalizer* and denoted $h^\star_{\rm nom}$.

When expressed in terms of the real and imaginary parts of the data, this problem is a second-order cone program (SOCP) [2]–[4]. Therefore, it can be readily solved using interior-point methods; see, *e.g.*, [2], [5], [6]. Semidefinite programming (SDP) techniques have also been used in the FIR filter design; see, *e.g.*, [7].

This equalizer design problem is illustrated in figure 1, where the equalizer H processes the output of the function G. The problem remains the same even if the order of the equalizer and given function is reversed. In this case, H is called a precoder.

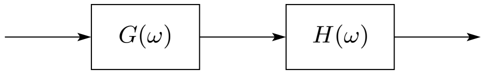

Fig. 1: Channel equalization.

B. Robust Chebyshev FIR equalization

The unequalized frequency response G is often not known exactly. As a representative example, when equalizing a communication channel, the unknown channel response is estimated by transmitting a known training sequence through the channel and then approximating G based on the received signal, so the estimated channel response is uncertain due to the imperfections in the statistical estimation procedure. Another example arises in audio applications, where we want to design an equalizer to improve sound quality in an uncertain environment, or at several physical locations simultaneously.

The performance of Chebyshev FIR equalization can be sensitive to the uncertainty or variation in the unequalized frequency response. It is often desired to account for this uncertainty when designing an equalizer (or precoder).

The goal of *robust equalization* is to choose filter coefficients h such that the equalized frequency response does not deviate much from the desired frequency response despite variations in the unequalized response $G_m, m = 1, \ldots, M$. We assume that each G_m is uncertain, but belongs to a known uncertainty set $\mathcal{G}_m$. In the worst-case robust optimization approach (with the uncertainty set described above), we judge the objective by its worst-case value over all possible data $G_m \in \mathcal{G}_m$,

$$C_{\mathrm{wc}}(h) = \max_{m} \sup_{G_m \in \mathcal{G}_m} \left| G_m w_m^* h - G_m^{\mathrm{des}} \right|,$$

which is called the worst-case (Chebyshev) maximum absolute deviation. The goal of the *robust Chebyshev FIR equalizer design* problem is to find the FIR filter that minimizes the worst-case maximum absolute deviation (over the given uncertainty model):

$$\text{minimize} \quad C_{\mathrm{wc}}(h). \tag{3}$$

Any solution of (3) is called a *robust optimal equalizer* and denoted $h^\star_{\mathrm{rob}}$.

We note that this problem is convex, since the objective is the pointwise supremum of a family of convex functions [8, Chap. 3]. However, it is not clear how to solve this problem directly, since the objective is given by a supremum over an infinite set.

We consider robust Chebyshev FIR equalization with an ellipsoidal model of uncertainty where the frequency response at each discrete frequency is uncertain but known to be inside an ellipse in the complex plane. The ellipse can be chosen by considering the distribution of the frequency response estimation error. As an example, when the error is complex Gaussian, we can take a confidence region as the uncertainty set. As another example, when the amplitude and phase of the error are independent and uniform distributions, the support of the error distribution becomes a disk. In this special case, we say that the uncertainty set is a disk in the complex plane.

The main goal of this paper is to show that the associated robust equalization problem (3) can be formulated as a semidefinite program (SDP), which interior-point algorithms can solve with great efficiency. (See, *e.g.*, [9] for more on semidefinite programming.) In the disk uncertainty case, the problem can be further simplified as a second-order cone program (SOCP), which interior-point algorithms can solve even more efficiently. (Here the computational effort to solve the robust equalization problem is about the same as for the nominal problem that ignores the uncertainty). When the uncertainty model is not given in the ellipsoidal form, we can approximate it with an ellipsoidal set and proceed with the given robust equalization approach.

The computational complexity of the nominal equalization problem is $O(Mn^2)$; in particular, it grows linearly with the number of frequency samples, and as the square of the order of the filter. In fact, the computational complexity of the robust equalization problem, with disk and general ellipsoidal uncertainty, is also $O(Mn^2)$, but with larger constants hidden in order notation.

C. Related work

Robust equalization has been studied in the literature since the 1980's [10], [11]. More recently, ideas from the (worst-case) robust optimization [12]–[14] have been applied to least-squares equalization [15, Sec. 3.4], minimax mean-squared-error (MSE) equalization [16], zero-forcing equalization (ZFE), minimum-mean-squared-error (MMSE) equalization, and MMSE with decision feedback equalization (MMSE-DFE) [17]. The previous work in robust equalization uses the least-squares metric to judge the approximation between the equalized response and the desired response.

Similar ideas of using the worst-case robust optimization have been successfully applied to related signal processing problems such as: robust filtering [18], [19], robust parameter estimation [20], [21], robust matched filtering [22], [23], robust minimum variance beamforming [24]–[28].

II. Robust equalization with ellipsoidal model

A. Ellipsoidal uncertainty model

We assume that the uncertainty in G_m is described by an ellipse in the complex plane:

$$G_m \in \mathcal{G}_m = \{ G_m^{\mathrm{nom}} + p_m u_1 + q_m u_2 \mid u_1^2 + u_2^2 \le 1 \}, \tag{4}$$

where $G_m^{\mathrm{nom}} \in \mathbf{C}$ is the nominal (estimated) value of the frequency response at frequency ω_m, p_m and $q_m \in \mathbf{C}$ are complex numbers that describe the shape of the uncertainty ellipse, and $u_1, u_2 \in \mathbf{R}$ are free parameters. Without loss of generality, we also assume that $q_m = i\mu p_m$, for some $\mu \in \mathbf{R}$. This is the same as saying that the angle between the complex numbers p_m and q_m is $90°$, *i.e.*, $\mathbf{Re}(p_m^* q_m) = 0$. We take $\mu \ge 1$, so p_m is the principal axis of the ellipse.

$$A_m(h) = \begin{bmatrix} \lambda_m & 0 & 0 & \mathbf{Re}\, p_m w_m^* h & \mathbf{Im}\, p_m w_m^* h \\ 0 & \lambda_m & 0 & \mathbf{Re}\, q_m w_m^* h & \mathbf{Im}\, q_m w_m^* h \\ 0 & 0 & t-\lambda_m & \mathbf{Re}\, G_m^{\rm nom} w_m^* h - G_m^{\rm des} & \mathbf{Im}\, G_m^{\rm nom} w_m^* h - G_m^{\rm des} \\ \mathbf{Re}\, p_m w_m^* h & \mathbf{Re}\, q_m w_m^* h & \mathbf{Re}\, G_m^{\rm nom} w_m^* h - G_m^{\rm des} & t & 0 \\ \mathbf{Im}\, p_m w_m^* h & \mathbf{Im}\, q_m w_m^* h & \mathbf{Im}\, G_m^{\rm nom} w_m^* h - G_m^{\rm des} & 0 & t \end{bmatrix} \in \mathbf{R}^{5\times 5}. \quad (5)$$

As a special case, when $\rho_m = |p_m| = |q_m|$, the ellipse $\mathcal{G}_m$ becomes a disk of radius ρ_m in the complex plane and can be equivalently expressed as

$$\mathcal{G}_m = \{G_m^{\rm nom} + \rho_m z \mid |z| \le 1\}, \quad (6)$$

where $z \in \mathbf{C}$ is a free parameter.

B. SDP formulation with ellipsoidal uncertainty

The main result of this paper is based on the fact that the robust Chebyshev FIR equalizer design problem (3) with the ellipsoidal uncertainty model (4) can be reformulated as the problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A_m(h) \succeq 0, \quad m = 1, \ldots, M, \end{array} \quad (7)$$

where the matrix $A_m(h) \in \mathbf{R}^{5\times 5}$ is given in (5). The optimization variables in the problem are $t \in \mathbf{R}$, $h \in \mathbf{R}^n$, and $\lambda_1, \ldots, \lambda_M \in \mathbf{R}$. The details of the equivalence between (3) and (7) when the uncertainty sets $\mathcal{G}_1, \ldots, \mathcal{G}_M$ are ellipsoidal are given in Appendix A.

The objective of problem (7) is to minimize a linear function over M linear matrix inequalities, which is a semidefinite program (SDP). Semidefinite programs are computationally tractable [9], and therefore the robust Chebyshev FIR equalization with the ellipsoidal uncertainty model is tractable. (Several high quality solvers for SDPs are available as open-source software, *e.g.*, SeDuMi [29], SDPT3 [30], and DSDP5 [31].)

We can see that the complexity is $O(Mn^2)$ as follows. First we note that interior-point methods almost always converge in a few tens of steps; in particular we can consider the number of steps as being constant [8]. In each step of an interior-point method, a set of linear equations in n variables has to be solved; the dominant cost is actually forming the coefficient matrix, not solving the resulting equations. This cost is $O(Mn^2)$ (using the fact that the LMIs have a fixed size, *i.e.*, 5×5).

C. SOCP formulation with disk uncertainty

With the disk uncertainty model in (6), we can find a simpler formulation of the robust Chebyshev FIR equalizer design problem (3) than the SDP (7). Specifically, with (6) we can reformulate (3) as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & |G_m^{\rm nom} w_m^* h - G_m^{\rm des}| + \rho_m |w_m^* h| \le t \\ & \qquad m = 1, \ldots, M, \end{array} \quad (8)$$

where the optimization variables are $t \in \mathbf{R}$ and $h \in \mathbf{R}^n$.

This problem is equivalent to an SOCP, when expressed in terms of the real and imaginary parts of the variables, which can be solved efficiently and globally using interior-point methods. The computational complexity is the same order, $O(Mn^2)$, as the nominal equalization problem (2) and the robust equalization problem (7), but with a constant that lies in between these two.

Next we show the equivalence between problem (3) with the disk uncertainty and the SOCP (8). We observe that we can introduce the epigraph variable t in problem (3) to obtain

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \sup_{G_m \in \mathcal{G}_m} \left|G_m w_m^* h - G_m^{\rm des}\right| \le t, \end{array}$$

where $m = 1, \ldots, M$. It suffices to show that

$$\sup_{G_m \in \mathcal{G}_m} \left|G_m H_m - G_m^{\rm des}\right| = \left|G_m^{\rm nom} H_m - G_m^{\rm des}\right| + \rho_m |H_m|,$$

where $H_m = w_m^* h$ and the supremum is taken over all $G_m \in \mathcal{G}_m$ given in (6). Using the triangle inequality, we have

$$\begin{aligned} \left|(G_m^{\rm nom} + \rho_m z) H_m - G_m^{\rm des}\right| &\le \left|G_m^{\rm nom} H_m - G_m^{\rm des}\right| + |\rho_m z H_m| \\ &= \left|G_m^{\rm nom} H_m - G_m^{\rm des}\right| + \rho_m |H_m|. \end{aligned}$$

Here the equality holds with the choice of $z = e^{i\phi}$ where the angle is

$$\phi = \angle\left(G_m^{\rm nom} H_m - G_m^{\rm des} - H_m\right).$$

This gives us the problem formulation (8), and completes the equivalence proof.

We give a simple interpretation of the relation

$$\sup_{G_m \in \mathcal{G}_m} \left|H_m G_m - G_m^{\rm des}\right| = \left|H_m G_m^{\rm nom} - G_m^{\rm des}\right| + \rho_m |H_m|.$$

We add to the original objective a weighted magnitude of the equalizer's frequency response that penalizes designs with large frequency response, which would amplify uncertainties in the unequalized response. Therefore, this robust design approach can be viewed as a *regularization* of the equalizer. (A similar observation is made in the context of robust beamforming with uncertain weights; see, *e.g.*, [28].)

III. Numerical Example

As a simple illustrative example, we design an FIR equalizer of length $n = 20$ for a channel with the nominal impulse response

$$g^{\rm nom} = 1/2\,(1/4, 1/2, 0, 1, 1/4).$$

and the nominal frequency response $G^{\rm nom}$ obtained by taking the Fourier transform of $g^{\rm nom}$. We sample the response at $M = 100$ frequencies $\omega_m = \pi(m-1)/M$, where $m = 1, \ldots, M$. We take the desired frequency response as $G^{\rm des} = e^{-iD\omega}$, where the delay is $D = 8$. The magnitude and the phase of the nominal frequency response $G^{\rm nom}$, together with the desired frequency response, are shown in figure 2.

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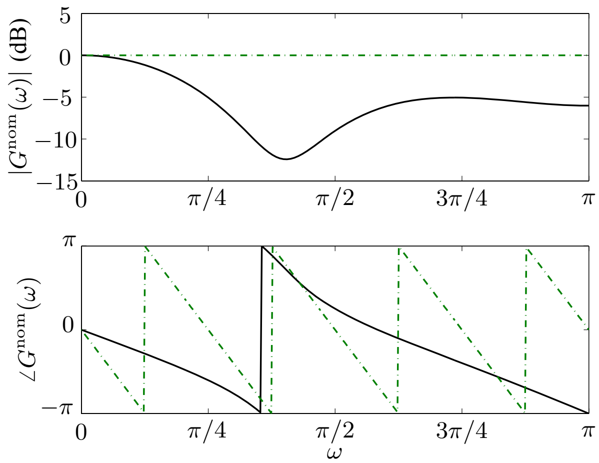

Fig. 2: Nominal frequency response of the channel (solid curve) versus the desired frequency response (dash-dotted curve). *Top.* Magnitude. *Bottom.* Phase.

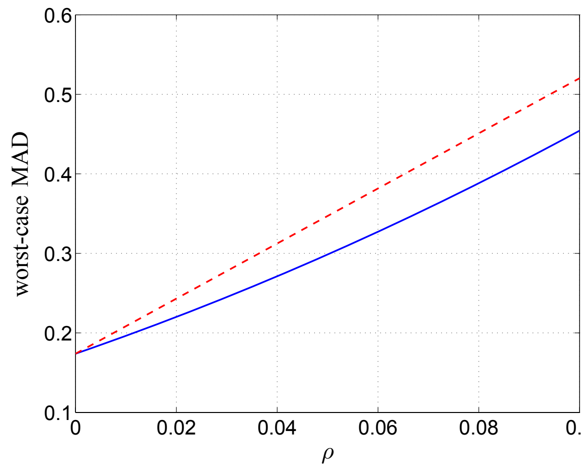

Fig. 3: Worst-case maximum absolute deviation (MAD) for the nominal $h^\star_{\rm nom}$ (dashed curve) and the robust optimal equalizer $h^\star_{\rm rob}$ (solid curve).

We consider ellipsoidal uncertainty in the channel frequency response G_m given by

$$\mathcal{G}_m = \{G_m^{\rm nom} + p_m u_1 + q_m u_2 \mid u_1^2 + u_2^2 \le 1\},$$

where the uncertainty radius ρ is equal for all response samples $m = 1, \ldots, M$.

We solve the nominal Chebyshev FIR equalization problem (2) and a family of robust equalization problems (8) for various values of ρ using the CVX software package [32], a Matlab-based modeling system for convex optimization. (The CVX package internally uses SDPT3 [30] as the solver.)

For a particular value of ρ, the worst-case maximum absolute deviation $C_{\rm wc}$ of an equalizer h is given by

$$C_{\rm wc}(h) = \max_{m=1,\ldots,M} \left(\left| G_m^{\rm nom} w_m^* h - G_m^{\rm des} \right| + \rho |w_m^* h| \right). \quad (9)$$

The worst-case maximum absolute deviation versus ρ is shown in figure 3. We note that the robust optimal equalizer performs better than the nominal one in the worst-case sense, *i.e.*, it always has a lower $C_{\rm wc}$ value.

We fix the value of uncertainty radius $\rho = 0.1$, and compute the nominal and robust optimal equalizers for this case. Figure 4 shows impulse responses of the nominal optimal and robust optimal equalizer. Figure 5 shows the nominal equalized response $H(\omega)G^{\rm nom}(\omega)$ (for the nominal and robust cases) and the desired response $G^{\rm des}(\omega)$. Figure 6 shows the worst-case equalized response $H(\omega)G^{\rm wc}(\omega)$ (for the nominal and robust cases) and the desired response $G^{\rm des}(\omega)$, where we have found the worst-case uncertainties for both the nominal optimal and robust optimal equalizers. The nominal optimal equalizer performs worse than the robust one given the uncertainty in the channel. In addition, the performance of the robust optimal equalizer does not degrade very much over all the possible channel realizations compatible with the uncertainty model.

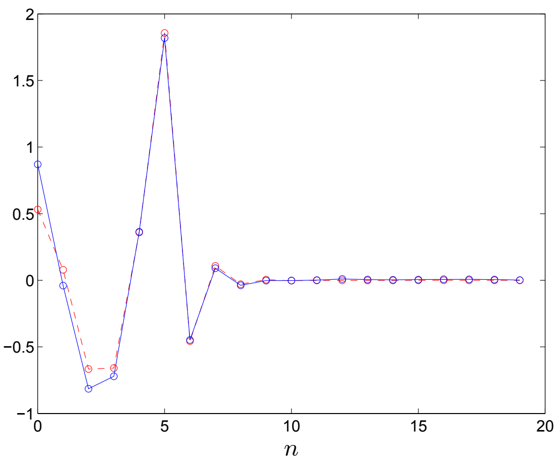

Fig. 4: Nominal (dashed curve) and robust (solid curve) equalizer impulse responses.

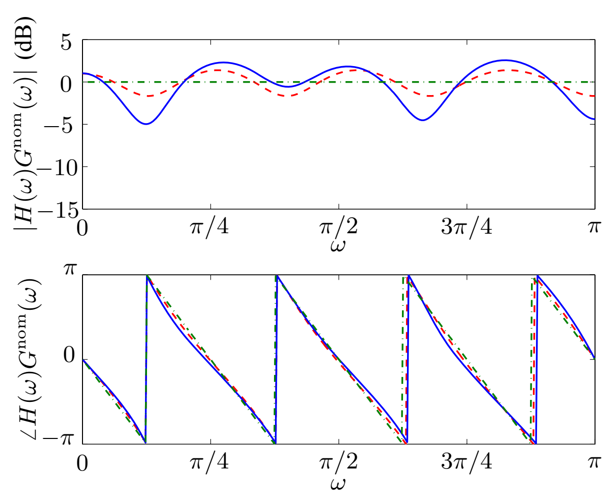

Fig. 5: Nominal and robust equalized transfer functions (dashed and solid curve, respectively) versus the ideal transfer function (dash-dotted curve) given the nominal channel. *Top.* Magnitude. *Bottom.* Phase.

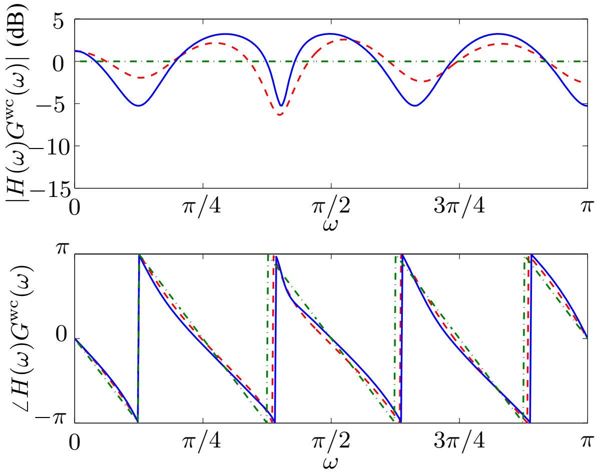

Fig. 6: Nominal and robust equalized transfer functions (dashed and solid curve, respectively) versus the ideal transfer function (dash-dotted curve) given the worst-case channel. *Top.* Magnitude. *Bottom.* Phase.

IV. Conclusions

In this paper, we have shown that the worst-case robust Chebyshev FIR equalization with ellipsoidal uncertainty in the unequalized frequency response can be cast as an SDP; in the case of disk uncertainty, the robust equalization problem can be cast as an SOCP. Both of these problems can be solved with $O(Mn^2)$ operations, which is the same order as the nominal equalization problem (which can also be posed an an SOCP). (The constants, however, are different.) The SOCP formulation (in the disk uncertainty case) can be interpreted as a weighted regularization of the nominal equalization problem.

The robust Chebyshev FIR equalization approach can be readily extended to robust filtering and equalization with complex FIR coefficients. Another straightforward extension is robust Chebyshev FIR equalization of multiple channels. Finally, the extension to robust Chebyshev FIR equalization with general nonseparable uncertainty models on the frequency responses of the unequalized channel appears to be very challenging. (In this paper, we have assumed that the frequency responses are subject to separable variations over the frequencies.)

Appendix

A. SDP formulation

We show the equivalence between problem (3) with an ellipsoidal uncertainty set and the SDP (7). We start by observing that we can introduce the epigraph variable t in (3) to obtain

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \sup_{G_m\in\mathcal{G}_m} \left|G_m w_m^* h - G_m^{\mathrm{des}}\right| \le t \\ & \qquad\qquad\qquad m = 1,\ldots,M, \end{array}$$

where the supremum is taken over all $G_m \in \mathcal{G}_m$ given in (4). The equivalence now follows directly from the following observation (which we will establish below):

$$\sup_{G_m\in\mathcal{G}_m} |G_m w_m^* h - G_m^{\mathrm{des}}| \le t$$

if and only if there exists $\lambda \in \mathbf{R}$ such that

$$\begin{bmatrix} \begin{matrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & t-\lambda \end{matrix} & F_m \\ F_m^T & tI \end{bmatrix} \succeq 0, \qquad (10)$$

where F_m is given by

$$F_m = \begin{bmatrix} \mathbf{Re}\, p_m w_m^* h & \mathbf{Im}\, p_m w_m^* h \\ \mathbf{Re}\, q_m w_m^* h & \mathbf{Im}\, q_m w_m^* h \\ \mathbf{Re}\, G_m^{\mathrm{nom}} w_m^* h - G_m^{\mathrm{des}} & \mathbf{Im}\, G_m^{\mathrm{nom}} w_m^* h - G_m^{\mathrm{des}} \end{bmatrix}.$$

We start by noting that

$$\sup_{G_m\in\mathcal{G}_m} \left|G_m w_m^* h - G_m^{\mathrm{des}}\right| \le t$$

if and only if $|G_m w_m^* h - G_m^{\mathrm{des}}| \le t$ for all $G_m \in \mathcal{G}_m$, *i.e.*, the following implication holds:

$$u_1^2 + u_2^2 \le 1 \implies |(G_m^{\mathrm{nom}} + p_m u_1 + q_m u_2) w_m^* h - G_m^{\mathrm{des}})| \le t. \qquad (11)$$

We introduce $v = (u_1, u_2, 1) \in \mathbf{R}^3$ to express $u_1^2 + u_2^2 \le 1$ as

$$v^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} v \le 0.$$

The right-hand side of the implication (11) is equivalent to

$$|(G_m^{\mathrm{nom}} + p_m u_1 + q_m u_2) w_m^* h - G_m^{\mathrm{des}})|^2 \le t^2,$$

which is in turn equivalent to

$$(F_m^T v)^T (F_m^T v) - t^2 \le 0.$$

Dividing the right-hand side of the implication by $t > 0$ and re-arranging terms, we obtain

$$v^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} v \le 0 \implies v^T \left(\frac{F_m F_m^T}{t} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t \end{bmatrix} \right) v \le 0.$$

Using the S-procedure [8, App. B], we can see that this implication between two quadratic forms is equivalent to the existence of $\lambda \in \mathbf{R}$ such that $\lambda \ge 0$ and

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - F_m (tI)^{-1} F_m^T + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t \end{bmatrix} \succeq 0.$$

Using the Schur complement technique [8, A.5.5], we can now see that the implication is equivalent to

$$\lambda \ge 0, \qquad \begin{bmatrix} \begin{matrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & t-\lambda \end{matrix} & F_m \\ F_m^T & tI \end{bmatrix} \succeq 0.$$

The condition $\lambda \ge 0$ is also enforced by the matrix inequality and thus can be dropped to obtain (10), which completes the proof.

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