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A Note on the Order of ℓ_1 Optimal Compensators

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Abstract— **Recently, Dahleh and Pearson [1] showed that when an ℓ_1 optimal compensator for a digital single-input, single-ouput, time-invariant plant exists, it is rational if the plant itself is. In this note, we give an example of a one-parameter family of *first order plants* where the order of the optimal compensator can be *arbitrarily* large. However, computational experience suggests that nearly optimal controllers of reasonable degree usually exist.**

I. OPTIMAL SENSITIVITY

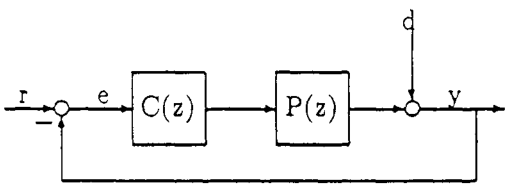

Figure 1: Basic Discrete Time Feedback System

Consider the discrete, linear, single-input, single-ouput, time-invariant feedback control system shown above, where r is a reference input and d is an additive (unknown) output disturbance. We shall denote the impulse response sequence from disturbance to output as $h_{yd}(k)$ and its transfer function as $H_{yd}(z)$ where

$$H_{yd}(z) = \sum_{k=0}^{\infty} h_{yd}(k) z^{-k}$$

Of course, from *Figure 1* we have $H_{yd} = \frac{1}{1+PC}$.

From approximately 1981 on, starting with Zames [2], and continuing with many workers including Helton [3], Chang [4], Francis [5], and Doyle [6], the H_∞ sensitivity minimization problem was formulated and gradually solved. The aim of the H_∞ problem is to pick the dynamic compensator, C, to stabilize the closed loop feedback system and to minimize:

$$\|H_{yd}\|_\infty \triangleq \sup_\omega |(H_{yd}(e^{j\omega})|$$

since this quantity gives the *energy* or ℓ_2 gain from disturbance to output. A compensator achieving this is refered to as an H_∞ optimal compensator.

Although H_∞ sensitivity has received much attention, and this methodology minimizes disturbance *energy* in the output, *it does not directly control the peak deviation.* Roughly speaking, $\| H_{yd} \|_\infty$ can be small but the closed loop can still greatly amplify peaks in the disturbance.

To address this shortcoming, a new sensitivity minimization problem was formulated by Vidyasagar [7], the ℓ_1 sensitivity minimization problem. Its aim is to pick the dynamic compensator, C, to stabilize the closed loop feedback system and to minimize:

$$\|H_{yd}\|_A = \sum_{k=0}^{\infty} |h_{yd}(k)| \triangleq \| h_{yd} \|_1$$

since this quantity gives the *peak* or ℓ_∞ gain from disturbance to output. A compensator achieving this aim is refered to as an ℓ_1 optimal compensator.

II. ℓ_1 SENSITIVITY

By employing a parameterization of all stabilizing controllers, the ℓ_1 sensitivity problem

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can be reduced to the following minimization problem:

$$\inf_{h \in \ell_1} \| f - g * h \|_1$$

$$f,\ g \text{ given sequences in } \ell_1$$

Where $*$ denotes convolution of sequences. (Recall that ℓ_1 is the space of all sequences h, with $\|h\|_1 < \infty$.) In [1], Dahleh and Pearson showed that any minimizer, h_{opt}, had the property that the optimal residual, $r_{opt} = f - g * h_{opt}$, was non-zero at only finitely many points. This implies that the optimal compensator is rational since

$$C_{opt}(z) = \frac{1 - R_{opt}(z)}{P(z) R_{opt}(z)}$$

A similar result is true for LQG or H_∞ optimal controllers as well. In these cases, moreover, we can bound the order of the optimal compensator in terms of the order of the plant. Indeed $2n + 1$, where n is the order of the plant, is always an upper bound on the order of these types of compensators. This is not the case with ℓ_1 optimal compensators and we give an example of a one-parameter family of *first order* plants whose ℓ_1 optimal compensators have *arbitrarily high order*. Thus we demonstrate that it is impossible to bound the order of the ℓ_1 optimal compensator in terms of the order of the plant. The example is as follows:

$$P_\epsilon(z) = \frac{1 + (2\epsilon - 1)z}{1 + (\epsilon - 1)z} \qquad 0 < \epsilon < \frac{1}{2}$$

The plant has an unstable pole at $\frac{1}{1-\epsilon}$ and a non-minimum phase zero at $\frac{1}{1-2\epsilon}$. Since P has no repeated non-minimum phase zeros or poles, and no complex non-minimum phase poles or zeros, we can apply the method of [1] for computing $C_{opt}(z)$. With this method, we solve the linear program:

maximize:

$$\alpha_1$$

subject to:

$$|(1 - 2\epsilon)^k \alpha_1 + (1 - \epsilon)^k \alpha_2| \le 1$$

$$k = 0, 1, 2, \ldots$$

The non-zero values of $r_{opt}(k)$ occur exactly at those k values that correspond to active constraints at the solution of the linear program above. By solving for the vertices of the linear polytope described by the constraints, it is seen that only the constraint corresponding to $k = 0$ and one constraint at $k = k^*$ will be active. In fact, for small epsilon one can show that

$$k^* = \arg\min \left[\frac{2 + e^{-\epsilon k} + e^{-2\epsilon k}}{e^{-\epsilon k} - e^{-2\epsilon k}} \right]$$

which gives $k \approx \frac{\sinh^{-1}(1)}{\epsilon}$ asymptotically as $\epsilon \longrightarrow 0$. The family of ℓ_1 optimal compensators corresponding to the given family of plants is:

$$C_\epsilon(z) = \left[\frac{(1-\epsilon)^{k^*} z^{k^*} - (1-\epsilon)^{k^*} + (1-2\epsilon)^{k^*}}{\left[(1-\epsilon)^{k^*} - (1-2\epsilon)^{k^*}\right] z^{k^*}} \right] P_\epsilon^{-1}(z)$$

which can be seen to be of order $k^* + 1$. For $\epsilon = .1$, the linear program was solved numerically and it was found that the order of $C_{opt}(z)$ was 10 —which is on the order predicted.

III. ACKNOWLEDGMENTS

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