$A(Z_1, Z_2)$ has no zeros as the unit hypercircle, however, for the optimal PLSI case the correctness of the modified Shanks' conjecture needs to be investigated.

V. Conclusions

A classification of LSI polynomials into optimal and suboptimal LSI polynomials is presented in this note. By an example it is shown that a suboptimal LSI polynomial can be unstable even if the original polynomial does not have any zeros on the unit circle. This has invalidated the proof for modified Shanks' conjecture in the 2-D case as presented in the paper[1] leaving the conjecture to remain a conjecture. We also hope that this note will clarify the well-known Robinson's result regarding the stabilization using least squares approach.

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Dynamical System State Need not Have Spectrum

STEPHEN BOYD AND LEON O. CHUA

Recently there has been much study of nonlinear dynamical systems (differential equations) which have ***chaotic solutions***. While there is not precise definition of what a chaotic solution is, it is generally agreed upon that a chaotic trajectory should have a ***continuous spectrum***, in particular, it should not be almost periodic [1]. A natural question is, therefore, *does a (bounded) trajectory of a dynamical system always have a spectrum*?. In this short note we give a simple example which shows that it need not.

What exactly do we mean by spectrum? In this case the appropriate definition of *spectrum* is that from *Generalized Harmonic Analysis* (GHA), introduced by Kolmogorov and Wiener in the 1930's to describe bounded signals which persist, that is, never fade. In GHA, the spectrum of a real-valued signal $u(t)$ is defined via its *autocorrelation* $R_u(\tau)$, given by

$$R_u(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T u(t)u(t+\tau)\,dt \qquad (1)$$

if the limit exists (it need not).[1] If $R_u(\cdot)$ is continuous at $\tau = 0$, then it can be represented as

$$R_u(\tau) = \int e^{i\nu\tau} S_u(d\nu)$$

where S_u is a positive bounded measure called the *spectral measure* of u. The spectrum of u is simply the support of S_u, roughly speaking, the set of ν's for which $S_u(d\nu)$ is nonzero. Those ν where S_u has a point mass are the *spectral lines* of u, and the rest of the spectrum is the *continuous spectrum* of u [2], [3].

Consider the dynamical system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 x_3 \\ x_1 x_3 \\ -x_3^2 \end{bmatrix} \qquad (2)$$

Note that the vector field is smooth, indeed it is quadratic.

The system (2) has a simple system-theoretic interpretation: x_1 and x_2 are the states of an oscillator, frequency modulated by x_3, which evolves according to $\dot{x}_3 = -x_3^2$. The general solution of (2) is easily found. For $x_3(0) \geq 0$, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} R\cos(\log(1 + tx_3(0)) + \delta) \\ R\sin(\log(1 + tx_3(0)) + \delta) \\ x_3(0)(1 + tx_3(0))^{-1} \end{bmatrix}$$

where $R = \sqrt{x_1^2(0) + x_2^2(0)}$ and $\delta = \tan^{-1}(x_1(0), x_2(0))$. Note in particular the trajectories are bounded when $x_3(0) \geqslant 0$.

If $x_3(0) > 0$, then the trajectory x has no autocorrelation, that is, the limit in (1) does not exist, and hence x has no spectrum. To simplify matters, we will demonstrate this for a specific initial condition. Suppose that $x(0) = [1,0,1]^T$, so that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos\log(1+t) \\ \sin\log(1+t) \\ (t+1)^{-1} \end{bmatrix}.$$

We will now show that x_1 has no average power, that is, $R_{x_1}(0)$

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[1] For vector valued u, the integrand is $u(t)u(t+\tau)^T$; if u is a sequence (1) is replaced in the obvious way by

$$\lim_{T \to \infty} T^{-1} \sum_{t=0}^{T-1} u(t)u(t+\tau)$$

where t, τ, and T are now integers.

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is not defined. Let

$$U_T = \frac{1}{T}\int_0^T x_1^2(t)\,dt.$$

Then U_T does not converge as $T \to \infty$, in fact

$$\limsup_{T\to\infty} U_T = \frac{1}{2}\left(1+\frac{1}{\sqrt{5}}\right) \qquad \liminf_{T\to\infty} U_T = \frac{1}{2}\left(1-\frac{1}{\sqrt{5}}\right). \quad (3)$$

Thus x_1 has no autocorrelation, and *a fortiori* x has no autocorrelation.

To establish (3) we simply note that

$$U_T = \frac{1}{2} + \frac{1}{10}(\cos 2\log(1+T) + 2\sin 2\log(1+T)) \quad (4a)$$

$$+ \frac{1}{10T}(\cos 2\log(1+T) + 2\sin 2\log(1+T) - 1). \quad (4b)$$

As $T \to \infty$, the term in (4b) converges to zero, and the expression in (4a) oscillates between the limits given in (3), which establishes our claim.

We close with a final comment. The skeptical reader may not accept that the GHA notion of spectrum, defined via the autocorrelation, is the only correct one here. That x does not have an autocorrelation is not simply a curious mathematical fact, but will manifest itself in purely operational terms, whenever an attempt is made to *measure* or *estimate* the spectrum of, say, x_1. For example, it is easy to verify that x_1 has no average or dc value, so that estimates of its average value made over larger and larger windows will not converge, but oscillate.

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Accurate Determination of Threshold Voltage Levels of a Schmitt Trigger

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Abstract—**In this paper the two threshold voltages of a Schmitt trigger are determined. It is shown that for most practical values of the feedback factor substantial errors can arise when one employs the commonly used expression for the width of the hysteresis region.**

Fig. 1 represents a Schmitt trigger built up with n-p-n transistors and Fig. 2 graphs the voltage transfer characteristic V_2 versus V_i. We assume base currents small enough to be fully neglected; further we can choose resistors R_a and R_b such that they do not create a substantial load to the collector of transistor T_1. By properly choosing m and the other parameters of the circuit, the transistors will be non-saturated, thus enabling us to use a one-sided Ebers–Moll model.

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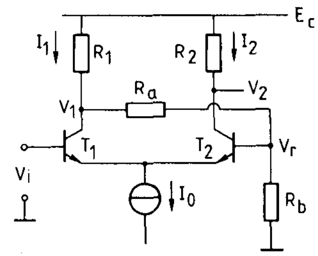

Fig. 1. A Schmitt-trigger circuit.

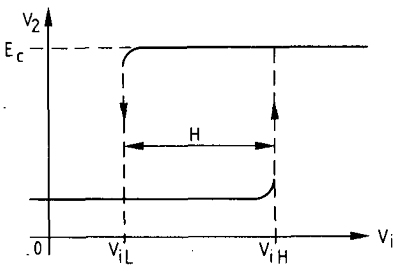

Fig. 2. Voltage transfer characteristic.

Let $V_r = V_1 \cdot R_b/(R_b + R_a) = m \cdot V_1$ and $I_1 = I_s \exp \varphi_{BE1}$; $I_2 = I_s \exp \varphi_{BE2}$, I_s is the saturation current of T_1 and T_2:

$$\varphi_n = V_n/V_T, \qquad V_T = kT/q \quad (1)$$

where we have also assumed $I_1, I_2 \gg I_s$.

With

$$I_1 + I_2 = I_0$$
$$I_1 = I_2 \cdot \exp(V_i - V_r)/V_T$$
$$V_r = m \cdot (E_C - I_1 R_1)$$
$$V_2 = E_C - I_2 R_2 \quad (2)$$

we can start to determine the two input threshold voltages V_{iL} and V_{iH}. To accomplish this we have to derive an expression for $D = dV_2/dV_i$ in order to let $D^{-1} = 0$.

We define $x = \exp(V_i - V_r)/V_T$ from which it follows:

$$\frac{dx}{dV_i} = \frac{x}{V_T}\left(1 - \frac{dV_r}{dV_i}\right).$$

With $I_1 = I_0(x/(x+1))$ it follows from (2):

$$\frac{dV_r}{dV_i} = 1 - \frac{V_T}{x}\cdot\frac{dx}{dV_i} = \frac{dV_r}{dx}\cdot\frac{dx}{dV_i}$$
$$= -mI_0R_1\frac{1}{(x+1)^2}\cdot\frac{dx}{dV_i}.$$

so

$$1 - \frac{V_T}{x}\cdot\frac{dx}{dV_i} = -m\frac{I_0R_1}{(x+1)^2}\cdot\frac{dx}{dV_i}$$

From (2) and $I_2 = I_0(1/(x+1))$ we find

$$\frac{dV_2}{dV_i} = \frac{dV_2}{dx}\cdot\frac{dx}{dV_i} = \frac{I_0R_1}{(x+1)^2}\cdot\frac{dx}{dV_i}.$$