Iterative Water-filling for Gaussian Vector Multiple Access Channels

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Abstract — **We develop an efficient iterative water-filling algorithm to find an optimal transmit spectrum for maximum sum capacity in a Gaussian multiple access channel with vector inputs and a vector output. The iterative algorithm converges from any starting point and reaches within $\frac{K-1}{2}$ nats per output dimension from the K-user sum capacity after just one iteration.**

I. Introduction

A discrete-time memoryless $K-$user Gaussian vector multiple access channel is modeled as:

$$\mathbf{y} = \sum_{i=1}^{K} H_i \mathbf{x}_i + \mathbf{n}, \tag{1}$$

where $\mathbf{x}_i$'s are vector inputs under power constraints P_i, i.e., $\mathbf{E}||\mathbf{x}_i||^2 \leq P_i$, $\mathbf{y}$ is the vector output, $\mathbf{n}$ is an i.i.d. Gaussian vector process with a covariance matrix Z, and H_i's are matrix channels. The capacity for this channel can be shown as [1] [2]:

$$\mathcal{C} = \bigcup_{p_1(\mathbf{x}_1)p_2(\mathbf{x}_2)} \left\{ (R_1, R_2) : \begin{array}{c} R_1 \leq I(\mathbf{x}_1; \mathbf{y}|\mathbf{x}_2); \\ R_2 \leq I(\mathbf{x}_2; \mathbf{y}|\mathbf{x}_1); \\ R_1 + R_2 \leq I(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}). \end{array} \right\},$$

where $p_1(\mathbf{x}_1)$ and $p_2(\mathbf{x}_2)$ are Gaussian distributions satisfying the power constraints. The usual convex hull operation is not needed because of the convexity of the capacity function and the constraint set. The boundary points on the capacity region may be characterized by maximizing a weighted rate sum $\sum_{i=1}^{K} \mu_i R_i$. Without loss of generality, let $\mu_K \geq \mu_{K-1} \geq \cdots \geq \mu_1$. Then, finding the boundary points is equivalent to solving the following convex programming problem [3] [4]:

$$\begin{array}{ll} \text{maximize} & \mu_1 \cdot \frac{1}{2} \log \left| \sum_{i=1}^{K} H_i S_i H_i^\tau + Z \right| - \mu_K \cdot \frac{1}{2} \log |Z| \\ & \sum_{j=2}^{K} (\mu_j - \mu_{j-1}) \cdot \frac{1}{2} \log \left| \sum_{i=j}^{K} H_i S_i H_i^\tau + Z \right| \\ \text{subject to} & \mathrm{tr}(S_i) \leq P_i, \quad i = 1, \cdots, K \\ & S_i \geq 0. \quad i = 1, \cdots, K \end{array}$$

where $|\cdot|$ denotes matrix determinant, and S_i is the covariance matrix for $\mathbf{x}_i$.

II. Main Results

The sum-capacity achieving input covariance matrices have the following characterization.

Theorem 1 *The set of input covariance matrices $\{S_i^*\}$ solves the rate-sum maximization problem*

$$\begin{array}{ll} \textit{maximize} & \frac{1}{2} \log \left| \sum_{i=1}^{K} H_i S_i H_i^\tau + Z \right| \\ \textit{subject to} & \mathrm{tr}(S_i) \leq P_i, \quad i = 1, \ldots, K \\ & S_i \geq 0, \quad i = 1, \ldots, K \end{array}$$

if and only if S_i^* is the single-user water-filling covariance matrix for the channel H_i with $\sum_{j=1, j\neq i}^{K} H_j S_j^* H_j^\tau + Z$ as noise, for all $i = 1, 2, \cdots K$, simultaneously.

It is trivial to see the only if part of the theorem. The if part depends on the fact that the objective function is convex and the constraints are separable for each user.

The simultaneous water-filling condition leads to an efficient algorithm for computing the rate-sum optimal input covariance matrices based on a sequence of single-user water-fillings.

Algorithm 1 *The iterative water-filling algorithm for a vector Gaussian multiple access channel works as follows:*

`initialize` $S_i = 0$, $i = 1, \ldots K$.
`repeat`
  `for` i`=1 to` K

$$N = \sum_{j=1, j\neq i}^{K} H_j S_j H_j^\tau + Z;$$

$$S_i = \arg\max_S \frac{1}{2} \log |H_i S H_i^\tau + N|;$$

  `end`
`until the desired accuracy is reached.`

Theorem 2 *The iterative water-filling algorithm converges to a limit point $\{S_i^*\}$ from any initial assignment of $\{S_i\}$. The limit point maximizes the sum rate of a K-user Gaussian vector multiple access channel. Furthermore, after just one iteration, $\{S_i\}$ achieves a sum data rate that is at most $(K-1)/2$ nats per output dimension away from the sum capacity.*

These results can be readily extended to fading channels and channels with ISI.

References

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