NUMERICAL METHODS FOR H_2 RELATED PROBLEMS

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Abstract

Recent results have shown that several H_2 and H_2-related problems can be formulated as convex programs with a *finite* number of variables. We present an interior point algorithm for the solution of these convex programs and illustrate its application with the standard LQR design.

1. Introduction

It has been shown recently that a number of H_2 and H_2-related problems can be formulated as convex programs with a finite number of variables — quadratic stabilization [1], mixed H_2/H_∞ and multicriterion LQG problems (see [2] and references therein). The common idea underlying these results is that though the original problem is not convex, a clever change of variables [3] makes it convex.

In this paper, we present a systematic procedure for transforming the convex programs resulting from H_2-related problems above into optimization over Affine Matrix Inequalities. We then present a simple interior point method for their solution. Though our presentation is through the simple LQR design example, the techniques readily extend to the more complicated problems cited above.

2. The LQR problem

Consider the linear time invariant system described by the state equations

$$\begin{aligned} \dot{x} &= Ax + Bu + w \\ z &= \begin{bmatrix} R^{\frac{1}{2}} & 0 \\ 0 & Q^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} \end{aligned} \qquad (1)$$

where u is the control input, w is unit intensity white noise and z is the output signal of interest. The LQR problem is to design a feedback controller from the state x to the control input u which minimizes the H_2 norm between w and z [4]. It is known that the optimal feedback law is a constant state feedback $u = -Kx$ and optimizes the following program:

$$\min_{P,K} \ \mathbf{Tr}(QP) + \mathbf{Tr}(R^{1/2}KPK^TR^{1/2})$$

subject to

$$(A - BK)P + P(A - BK)^T + I < 0 \text{ and } P = P^T > 0.$$

By defining a new quantity $Y = KP$, the above problem can be written as

$$\min_{P,Y} \ \mathbf{Tr}(QP) + \mathbf{Tr}(R^{1/2}YP^{-1}Y^TR^{1/2}) \qquad (2)$$

subject to

$$AP + PA^T - BY - Y^TB^T + I < 0 \text{ and } P = P^T > 0,$$

which is a convex program (see, for example, [1], [2]).

3. Transformation to Optimization over AMI's

The general structure of the convex program (2) is

$$\min_{(P,Y)\in\mathcal{A}} J(P,Y)$$

where $\mathcal{A}$ is some convex constraint set, and J a performance index. We show how this can be transformed into the problem

$$\min_{C(\gamma,Z)>0} \gamma \qquad (3)$$

where (γ, Z) is a new set of variables and C is symmetric and an affine matrix function of (γ, Z). The inequality $C(z) > 0$ is called an *Affine Matrix Inequality* (AMI).

Example: The LQR problem

The objective function of program (2) consists of the sum of two terms. It is easily shown that the second term

$$\Phi(P,Y) = \mathbf{Tr}(R^{1/2}YP^{-1}Y^TR^{1/2})$$

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can be expressed as

$$\Phi(P,Y) = \min(\mathbf{Tr}(X))$$
$$\begin{bmatrix} X & R^{1/2}Y \\ Y^T R^{1/2} & P \end{bmatrix} > 0.$$

Then let

$$C_1(\gamma,P,Y,X) := -\mathbf{Tr}(QP) - \mathbf{Tr}(X) + \gamma,$$
$$C_2(\gamma,P,Y,X) := -AP - PA^T + BY + Y^T B^T - I,$$
$$C_3(\gamma,P,Y,X) := \begin{bmatrix} X & R^{1/2}Y \\ Y^T R^{1/2} & P \end{bmatrix},$$
$$C(\gamma,P,Y,X) := \mathrm{diag}(C1,C2,C3).$$

The optimization problem (2) can now be written

$$\begin{array}{l} \text{minimize } \gamma \\ C(\gamma,P,Y,X) > 0 \end{array} \qquad (4)$$

which indeed is of the form (3).

Well-Posedness

We say that the convex program (3) is well-posed if for every real γ the set $\{Z \mid C(\gamma,Z) > 0\}$ is compact or empty. We observe the following without proof:

Proposition 1: *The program (4) corresponding to the LQR problem is well-posed if R is positive definite, (A,B) is controllable and (Q,A) is observable.*

Under similar assumptions, all the other problems cited in the introduction enjoy the same property.

4. Computational aspects

Problem (3) is a convex non-differentiable optimization program, and the ellipsoid algorithm or Kelley's cutting-plane algorithm [4] may be used to solve it. Recently, the work of Nemirovski *et al.* has led to the development of interior point algorithms based on the notion of the *analytic center* for a set of convex constraints [5]; these algorithms seem to hold great promise.

We will describe one such interior point algorithm, called the *method of centers.* Given an initial feasible point (γ_u, Z_u) for constraint C in program (3), and a desired absolute accuracy ϵ on the optimum, the algorithm is as follows:

while $\gamma_u - \gamma_l > \epsilon$,
  $\gamma_0 := \gamma_u + \epsilon$,
  $(\gamma^\star, Z^\star) :=$ a_center $(\mathrm{diag}(C, \gamma_0 - \gamma) > 0)$,
  $\gamma_u := \gamma^\star$,
  $Z_u := Z^\star$,
  Compute a lower bound γ_l.
end

Remark 1: An initial lower bound γ_l can be chosen to be 0.

Remark 2: Computing the analytic center of a convex bounded set (a_center($\mathrm{diag}(C, \gamma_0 - \gamma) > 0$)) needs an initial point interior to the constraint. (γ_u, Z_u) is such a point.

Remark 3: The lower bound γ_l is computed from $\gamma^\star$ and the Hessian of the barrier function of the constraint expressed at $(\gamma^\star, Z^\star)$. For more information, see [5], [6] and also [7] (these proceedings).

5. Conclusion

Through the LQR example, we have outlined a systematic procedure for transforming convex optimization programs arising from H_2-related problems into optimization over AMI's. We have also briefly described a simple interior point method for their solution. Our procedure easily applies to problems in [1] and [2], and more generally to many quadratic Lyapunov function shaping problems.

References

[1] P.L.D. Peres, J.C. Geromel, and S.R. Souza, Optimal $\mathbf{H}_2$ control for uncertain systems, In *Proc. American Control Conf.*, Chicago, June 1992.

[2] Pramod P. Khargonekar and Mario A. Rotea, Mixed $\mathbf{H}_2/\mathbf{H}_\infty$ control: a convex optimization approach, *IEEE Trans. Aut. Control*, **36**(7):824–837, July 1991.

[3] Alberto R. Galimidi and B. Ross Barmish, The constrained Lyapunov problem and its application to robust output feedback stabilization, *IEEE Trans. Aut. Control*, **AC-31**(5):410–419, May 1986.

[4] S. Boyd and C. Barratt, *Linear Controller Design: Limits of Performance*, Prentice-Hall, 1991.

[5] Yu. E. Nesterov and A. S. Nemirovsky, *Interior point polynomial methods in convex programming: Theory and applications*, Lecture notes in mathematics. Springer Verlag, 1992.

[6] S. Boyd and L. El Ghaoui, Method of centers for minimizing generalized eigenvalues, *in preparation for special issue of LAA on Numerical Linear Algebra Methods in Control, Signals and Systems*, 1992.

[7] V. Balakrishnan, E. Feron, S. Boyd, and L. El Ghaoui, Computing bounds for the structured singular value via an interior point algorithm, In *Proc. American Control Conf.*, Chicago, June 1992.