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Robust Control Design for Ellipsoidal Plant Set

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Abstract This paper presents a control design method for continuous-time plants whose uncertain parameters in the output matrix are only known to lie in an ellipsoidal set. The desired control is chosen to minimize the maximum linear quadratic regulator (LQR) cost from all the plants with parameters in the given set. Although no particular form is assumed for the minimax control, it turns out that it is the LQR control for one of the plants in the set, the worst-case plant. By defining an appropriate mapping, which maps an element from the given ellipsoidal set to an element of the same set, the existence of this worst-case plant is proved. A simple heuristic algorithm used to compute the worst-case plant is also given.

1 Introduction

A problem of great interest in control theory is the design of a controller which can guarantee some level of performance in the presence of plant parameter uncertainty. Kharitonov's theorem provides a necessary and sufficient analysis test for determining the robust stability of polynomials with perturbed coefficients, however, there are few results that exploit Kharitonov's theorem for synthesizing robust controllers, *e.g.*, [7] and [12]. Another approach to this problem is to define a set of nominal values of the uncertain parameters and consider deviations from these nominal values. A comprehensive survey of the different parameter space methods for robust control design, as opposed to frequency domain methods, can be found in [23].

The technique of solving control problems as minimax optimization problems is the basis of the so-called "$\mathbf{H}_\infty$ optimal control theory." In the standard $\mathbf{H}_\infty$ problem, the control input is chosen to minimize the norm of the output and the exogenous input is chosen to maximize it [2]. Along this line, the structured singular value (μ) synthesis method is used to find controllers which minimize a $\mathbf{H}_\infty$ objective subject to plant perturbations, *e.g.*, see [8], [9], and references therein. In [20], a game theoretic approach is used, where the control, restricted to a function of the measurement history, plays against adversaries composed of the process and measurement disturbances, and the initial states. Another example of solving control problems as minimax problems is [18], which presents a controller design method to minimize the weighted sum of the maximum linear quadratic gaussian (LQG) performance objectives over a set of worst plant parameter changes.

The approach of using set-membership to describe plant parameter uncertainty has gained popularity in recent years, *e.g.*, [14], [16], [26], [3], [17], and references therein. This approach of parameter identification is originated from early works of [22] and [5], where the set of possible system states compatible with the observations is shown to be an ellipsoid. Motivated by ellipsoidal bounds on plant parameters, we pose the following robust control problem: given that the unknown parameters in the output matrix of the plant are known to lie in an ellipsoid, find the control which minimizes the maximum LQR cost from all plants with parameters in the given set. Viewed in terms of game theory, the control and plant uncertainty are strategies employed by opposing players in a game, where the control is chosen to minimize the LQR cost and the plant uncertainty is chosen to maximize it. As a special case of our problem, finding the finite-horizon control for a discrete finite-impulse response (FIR) plant, was solved in [15]. In that case, it was shown that the minimization is a convex optimization problem. In this paper, we are generalizing the robust control design problem to find the infinite-horizon controls for continuous plants.

The assumption that the output matrix in the plant description contains all the uncertainty deserves further discussion. First, this is a natural extension of the discrete FIR finite-horizon problem solved in [15]. In the discrete case, FIR model sets can be identified from input-output data of a plant, *i.e.*, the coefficients of the FIR model are identified to belong to a set. This is particularly attractive when a bounded noise model, often a more realistic assumption than a statistical noise model, is used in the identification [19]. In the continuous case, Laguerre models can be used so that the identification is reduced to estimating the Laguerre coefficients [25]. Uncertainty in the Laguerre coefficients can then be described by set membership of the output matrix. Second, by limiting uncertain parameters to the output matrix, we simplify the analysis and gain more insights into the nature of the solution.

The paper is organized as follows, after stating the

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problem in the next section, the minimax control is proved in section 3 to be the LQR control designed for the worst-case plant from the given ellipsoidal set. By defining an appropriate mapping, which maps an element of the given set to an element of the same set, the existence of this worst-case plant is proved. In section 4, a simple algorithm used to compute the worst-case plant is given. A two-mass-one-spring example is used in section 5 to illustrate the ideas presented. The paper concludes with some remarks in section 6.

2 Problem Formulation

Consider the following family of systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \quad x(0) = x_0 & (1) \\ y(t) &= c^T x(t), & (2) \end{aligned}$$

where A, b, and x_0 are fixed and given, and

$$c \in \Theta = \left\{ \theta : (\theta - \theta_c)^T R (\theta - \theta_c) \leq 1, \quad R = R^T > 0 \right\}. \quad (3)$$

For a given control, $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, and a fixed $c \in \Theta$, the LQR cost is defined to be

$$J(u, c) \triangleq \int_0^\infty \left[r u(t)^2 + y(t)^2 \right] dt. \quad (4)$$

We assume that $(A,\ b)$ is controllable (or at least stabilizable) and $(c,\ A)$ is observable (or at least detectable) for all c in Θ. The robust control design problem is to find a control $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ that solves the following minimax problem:

$$\min_u \max_{c \in \Theta} J(u, c). \quad (5)$$

Since no particular form is assumed for the control u, such as linear state-feedback, the minimization in (5) is over all possible u's. Note also that we chose the initial time $t = 0$ for convenience only, the problem can be posed at any initial time $t = t_0$. Therefore, one can design a new controller each time Θ gets updated.

The cost objective in (4) and the ellipsoidal set in (3) lead to another interesting interpretation for the minimax problem in (5) once we rewrite (4) as

$$J(u, c) = \int_0^\infty \left[r u(t)^2 + x^T(t) c c^T x(t) \right] dt. \quad (6)$$

Now, instead of saying that we are designing a controller for a set of uncertain plants described by (1) through (3), we can also say that we are designing a controller for a set of uncertain objective functions. (This interpretation contrasts with the standard LQR design where a controller is obtained for fixed weighting matrices.) Note that $c^T x(t)$ is a dot product, so it depends on the angle between c and $x(t)$. Geometrically, the set Θ sweeps out a "cone" (with a curved base) of possible c's. Thus, we can interpret Θ as a set of "view angles" from which we calculate the cost. The minimax control from (5) is therefore robust to all these "view angles." This interpretation is interesting since in practice we seldom look at performance from just one angle.

3 Minimax Solution

To solve the minimax problem in (5), recall from [6, pages 274-282] that (u^*, c^*) is a saddle point if

$$J(u^*, c) \leq J(u^*, c^*) \leq J(u, c^*) \quad (7)$$

for all $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $c \in \Theta$. In that case, we have

$$(u^*, c^*) = \arg \min_u \max_{c \in \Theta} J(u, c) = \arg \max_{c \in \Theta} \min_u J(u, c). \quad (8)$$

Our goal in this section is to prove that there always exists such (u^*, c^*) for (5).

From LQR control theory, the second inequality in (7) is true if

$$u^* = u_{LQR}(c^*), \quad (9)$$

where $u_{LQR}(c^*)$ denotes the LQR control designed for the plant in (1) with $c = c^*$ in (2). It follows that the first inequality in (7) is also true if

$$c^* = \arg \max_c J\left(u_{LQR}(c^*), c\right). \quad (10)$$

Thus, if c^* exists for (10), the minimax problem in (5) is solved by (9). Note that the existence of c^* is not obvious because c^* must have the property that when $u_{LQR}(c^*)$ is applied to each $c \in \Theta$, the maximum cost occurs at c^*.

We now express the LQR cost in (10) in a more convenient form. Since $(A,\ b)$ is stabilizable and $(c,\ A)$ is detectable for all c in Θ, for each $c \in \Theta$ there is an associated state-feedback control $u_{LQR}(c)$ given by

$$u_{LQR}(c) = -K_c x, \quad (11)$$

where

$$K_c = \frac{1}{r} b^T P_c \quad (12)$$

and P_c satisfies the algebraic Riccati equation

$$A^T P_c + P_c A - \frac{1}{r} P_c b b^T P_c + c c^T = 0. \quad (13)$$

We will use X_c to denote the solution of the associated Lyapunov equation,

$$(A - bK_c) X_c + X_c (A - bK_c)^T + x_0 x_0^T = 0, \quad (14)$$

where

$$X_c = \int_0^\infty e^{(A - bK_c)t} x_0 x_0^T e^{(A - bK_c)^T t} \, dt. \quad (15)$$

The LQR cost in (10) can now be expressed as

$$\begin{aligned} & J(u_{LQR}(c^*), c) \\ &= \int_0^\infty \left[r u_{LQR}(c^*)^2 + y^2 \right] dt \\ &= \int_0^\infty \left[r K_{c^*} x x^T K_{c^*}^T + c^T x x^T c \right] dt \\ &= \int_0^\infty r K_{c^*} e^{(A-bK_{c^*})t} x_0 x_0^T e^{(A-bK_{c^*})^T t} K_{c^*}^T \, dt \\ &\quad + \int_0^\infty c^T e^{(A-bK_{c^*})t} x_0 x_0^T e^{(A-bK_{c^*})^T t} c \, dt \\ &= r K_{c^*} X_{c^*} K_{c^*}^T + c^T X_{c^*} c. \end{aligned} \tag{16}$$

For a given c^*, $K_{c^*} X_{c^*} K_{c^*}^T$ in (16) is fixed. Thus, the maximization in (10) becomes

$$c^* = \arg\max_c c^T X_{c^*} c. \tag{17}$$

Note that the feedback gain K_{c^*} does not depend on the initial condition x_0, but the Lyapunov solution X_{c^*} does. Therefore, the solution c^* is a function of x_0. However, this dependence on x_0 can be removed if we start with the assumption that x_0 is a random vector with known mean m and covariance C and the objective in (4) is an expectation over x_0. In that case, X_{c^*} is the solution of (14) with $x_0 x_0^T$ replaced by $C + mm^T$.

Our ultimate goal is to find c^* in (17), but we must first prove that such c^* always exists. To do that, we define the mapping $f: \bar{c} \in \Theta \to \hat{c} \in \Theta$,

$$\begin{aligned} f(\bar{c}) &= \hat{c} \\ &\triangleq \arg\max_c c^T X_{\bar{c}} c, \end{aligned} \tag{18}$$

where $X_{\bar{c}}$ satisfies the Lyapunov equation associated with $\bar{c}$ as in (14). It was shown in [15] that the solution of (18) is given by

$$\hat{c} = T\Lambda^{-\frac{1}{2}} \hat{z} + \theta_c \in \Theta_b, \tag{19}$$

where

$$R = T\Lambda T^T \tag{20}$$

$$\hat{z} = (\Omega - \hat{\lambda} I)^{-1} \beta \tag{21}$$

$$\hat{\lambda} = \max \lambda \left(\begin{bmatrix} \Omega & -I \\ -\beta\beta^T & \Omega \end{bmatrix} \right) \tag{22}$$

$$\Omega = R^{-\frac{T}{2}} X_{\bar{c}} R^{-\frac{1}{2}} \tag{23}$$

$$\beta = -R^{-\frac{T}{2}} X_{\bar{c}} \theta_c \tag{24}$$

$$\Theta_b = \{\theta : (\theta - \theta_c)^T R (\theta - \theta_c) = 1\} \tag{25}$$

(Θ_b is the boundary of Θ.) Therefore, the mapping f consists of two parts. First, it takes the given $\bar{c}$ and produces $X_{\bar{c}}$ via equations (12) through (14). Then $\hat{c}$ is given by (19).

To show that c^* exists in (17) is equivalent to showing that a fixed point c^* exists for f, *i.e.*,

$$f(c^*) = c^*. \tag{26}$$

To do that, we need a lemma extracted from [11] and a simple form of Brouwer's Theorem [13, pages 366-367].

Lemma 1 *If $(A,\ b)$ is stabilizable, then over any region where $(c,\ A)$ is detectable, the algebraic Riccati equation solution P_c in (13) is continuous in cc^T.*

Proof of Lemma 1 Consider the matrix-valued functional

$$g\left(P, cc^T\right) = A^T P + PA - \frac{1}{r} P b b^T P + cc^T. \tag{27}$$

For any c, P_c satisfies (13), so $g\left(P_c, cc^T\right) = 0$. As a quadratic function in P and a linear function in cc^T, the functional g is infinitely differentiable, and its derivative with respect to P at the point $\left(P_c, cc^T\right)$ is the linear operator given for any matrix Z by

$$Dg_P(Z) = (A - bK_c)^T Z + Z(A - bK_c). \tag{28}$$

Since K_c is stabilizing, the operator Dg_P is nonsingular by Lyapunov's equation. Therefore, from the implicit function theorem (see, *e.g.*, [21, pages 375-380]), there exists an infinitely differentiable matrix-valued function Ψ such that

$$P_c = \Psi(cc^T). \tag{29}$$

Thus, P_c is continuous in cc^T. □

Theorem 2 (Brouwer's Theorem) *Let C be a compact, convex subset of $\mathbb{R}^n$. Then any continuous function $f: C \to C$ has at least one point c^* such that $f(c^*) = c^*$.*

The existence of c^* in (17) can now be guaranteed by the following theorem.

Theorem 3 (Fixed Point) *The mapping f defined in (18) is continuous in $\bar{c}$ and it has a fixed point.*

Proof of Theorem 3 First, we need to show that the mapping from $\bar{c}$ to $X_{\bar{c}}$ is continuous.

1. Let $c = \bar{c}$ in (12) through (14). By Lemma 1, $P_{\bar{c}}$ of (13) is continuous in $\bar{c}\bar{c}^T$. Since each element of $\bar{c}\bar{c}^T$ is simply a product of elements from $\bar{c}$, $\bar{c}\bar{c}^T$ is continuous in $\bar{c}$. By the continuity of composite functions, $P_{\bar{c}}$ is continuous in $\bar{c}$.

2. $K_{\bar{c}}$ of (12) is continuous in $P_{\bar{c}}$, thus it is continuous in $\bar{c}$.

3. By the implicit function theorem (similar to the proof of Lemma 1), $X_{\bar{c}}$ is continuous in $K_{\bar{c}}$. By the continuity of composite functions, $X_{\bar{c}}$ is continuous in $\bar{c}$.

Second, we need to show that the mapping from $X_{\bar{c}}$ to $\hat{c}$ is also continuous.

1. Both Ω and β in (23) and (24) are continuous in $X_{\bar{c}}$. Since each eigenvalue of a matrix is continuous in the elements of the matrix (see, *e.g.*, [10, pages 191-192]), $\hat{\lambda}$ in (22) is continuous in $X_{\bar{c}}$. Thus, by the continuity of composite functions, $\hat{\lambda}$ is continuous in $\bar{c}$.

2. Each element of $(\Omega - \hat{\lambda}I)^{-1}$ is given by its cofactor divided by $\det(\Omega - \hat{\lambda}I)$. The cofactors and $\det(\Omega - \hat{\lambda}I)$ are sums of products of elements of $\Omega - \hat{\lambda}I$. Thus, $(\Omega - \hat{\lambda}I)^{-1}$ is continuous in $\bar{c}$, which implies $\hat{z}$ in (21) is continuous in $\bar{c}$ also. (Exception is when $\Omega - \hat{\lambda}I$ is singular, which is treated in [15]. However, continuity is not affected.)

3. $\hat{c}$ in (19) is continuous in $\bar{c}$.

Therefore, the mapping f from $\bar{c}$ to $\hat{c}$ is continuous, and by Brouwer's Theorem it has at least one fixed point. □

The existence of a saddle-point solution for the minimax problem in (5) is stated in the following theorem.

Theorem 4 (Existence) *There exists at least one (u^*, c^*) such that (7) is true and the minimax problem in (5) has a saddle-point solution. If there are more than one (u, c) which satisfy (7), then their associated LQR costs must be equal and any one of the solutions is equally valid.*

Proof of Theorem 4 From Theorem 3, we know that (10) has at least one fixed point. Therefore, (7) has at least one saddle-point solution. To show that two fixed points of (10) must have the same LQR cost, assume that there exist (u_1, c_1) and (u_2, c_2) such that

$$J(u_1, c) \leq J(u_1, c_1) \leq J(u, c_1), \quad \forall\, u,\ c \tag{30}$$

and

$$J(u_2, c) \leq J(u_2, c_2) \leq J(u, c_2), \quad \forall\, u,\ c. \tag{31}$$

Then let $c = c_2$ and $u = u_2$ in (30), we get

$$J(u_1, c_2) \leq J(u_1, c_1) \leq J(u_2, c_1) \leq J(u_2, c_2) \tag{32}$$

or

$$J(u_1, c_1) \leq J(u_2, c_2). \tag{33}$$

Similarly, let $c = c_1$ and $u = u_1$ in (31), we get

$$J(u_2, c_1) \leq J(u_2, c_2) \leq J(u_1, c_2) \leq J(u_1, c_1) \tag{34}$$

or

$$J(u_2, c_2) \leq J(u_1, c_1). \tag{35}$$

Therefore, (33) and (35) imply

$$J(u_1, c_1) = J(u_2, c_2). \tag{36}$$

□

This section can be summarized as follows: a fixed-point solution c^* exists for (10) and the solution to the minimax problem in (5) is given by (9). We now turn to the computation of c^*.

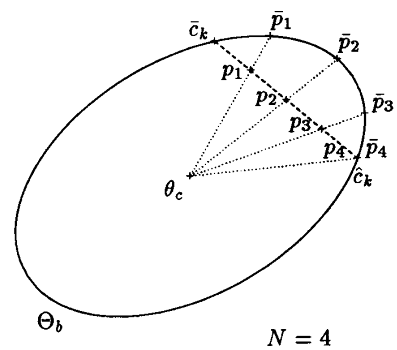

Figure 1: Candidate points used in calculating $\bar{c}_{k+1}$ and $\hat{c}_{k+1}$.

4 Fixed-Point Computation

Before describing our simple heuristic algorithm, we should point out that there exist many algorithms to compute Brouwer fixed points (see *e.g.*, [1] and [24].) Although these algorithms can guarantee that the fixed points will be found, they are known to have combinatorial complexity. In comparison, we have no guarantee that our algorithm will converge, but in many cases that we have tried, it usually converges in less than 10 iterations.

The goal of the iterative algorithm below is to find $\bar{c}_k$ such that the distance between $\bar{c}_k$ and $\hat{c}_k = f(\bar{c}_k)$, as defined in (18), is small, *i.e.*, a fixed point. Given $\bar{c}_k$ and $\hat{c}_k$ at the kth iteration, steps 6 through 8 below are designed to find $\bar{c}_{k+1}$ and $\hat{c}_{k+1}$. The algorithm accomplishes this by doing a local minimization over a set of candidate points, $\{\bar{p}_i,\ i = 1, \ldots, N\}$. Let $\{p_i,\ i = 1, \ldots, N\}$ be $N - 1$ equally-spaced points between $\bar{c}_k$ and $\hat{c}_k$, with $p_N = \hat{c}_k$ (see Figure 1). Vectors are then drawn from θ_c to each p_i, until they intersect Θ_b at points $\{\bar{p}_i,\ i = 1, \ldots, N\}$, where

$$\begin{aligned} \bar{p}_i &= \gamma w + \theta_c & (37) \\ \gamma &= \sqrt{\frac{1}{w^T R w}} & (38) \\ w &= \frac{p_i - \theta_c}{\|p_i - \theta_c\|_2}. & (39) \end{aligned}$$

Next, we compute $\hat{p}_i = f(\bar{p}_i)$ in step 7. After comparing the distances $\|\hat{p}_i - \bar{p}_i\|_2$, the $\bar{p}_j$ and $\hat{p}_j$ with the minimum distance become $\bar{c}_{k+1}$ and $\hat{c}_{k+1}$, respectively.

A Heuristic Algorithm

1. Define the mapping f from $\bar{c}$ to $\hat{c}$: compute $X_{\bar{c}}$ in (18) using (13), (12), and (14) then compute $\hat{c}$ using (19);
2. $k \leftarrow 0$;
3. Let $\bar{c}_1$ be a random point on Θ_b;
4. Compute $\hat{c}_1 = f(\bar{c}_1)$;
5. $k \leftarrow k+1$;
6. Compute $\{\bar{p}_i,\ i = 1,\ldots,N\}$ on Θ_b using (37);
7. Compute $\hat{p}_i = f(\bar{p}_i)$ for $i = 1,\ldots,N$;
8. Compute
$$j = \arg\min_i \|\hat{p}_i - \bar{p}_i\|_2 \tag{40}$$
then
$$\bar{c}_{k+1} = \bar{p}_j \tag{41}$$
$$\hat{c}_{k+1} = \hat{p}_j; \tag{42}$$
9. If $\|\hat{c}_{k+1} - \bar{c}_{k+1}\|_2 > \epsilon$, go to step 5.

Note that there is no guarantee that $\|\hat{c}_k - \bar{c}_k\|_2 < \|\hat{c}_{k+1} - \bar{c}_{k+1}\|_2$, so we don't have a convergence proof for this algorithm. However, with $\epsilon = 0.001$, this algorithm usually converges in less than 10 iterations.

5 Example

We will use the two-mass-one-spring system described in [4] in our example. This system, shown in Figure 2, can be represented in state-space form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ \frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u \tag{43}$$

$$y = c^T x \tag{44}$$

where x_1 and x_2 are the positions of masses 1 and 2, and x_3 and x_4 are the velocities of masses 1 and 2, respectively. We use masses $m_1 = m_2 = 1$ kg and spring coefficient $k = 1$ N/m for this system.

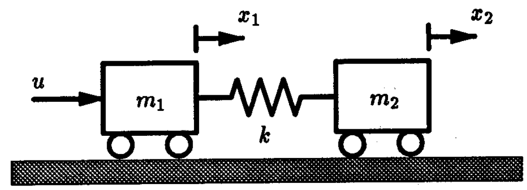

Figure 2: Two-mass-one-spring system.

The initial condition is $x_0 = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T$, which means the masses are displaced toward each other. For the ellipsoidal set in (3), we use $\theta_c = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T$ and $R = I$. Thus, the output y is nominally the sum of the position and velocity of the second mass, but c can still be anywhere within the unit ball. We choose $r = 1$ in the objective and $N = 4$ in the fixed-point algorithm. For the stopping criterion, $\epsilon = 0.001$ is used. The algorithm converges in 5 iterations.

Table 1 shows the cost matrix for this example, where c_{LQR} is the element in Θ which maximizes the cost for $u = u_{LQR}(\theta_c)$. As expected, the control $u = u_{LQR}(\theta_c)$ applied to $c = \theta_c$ gives the lowest cost for this control, 5.6, but its cost can be quite high at other c's such as c_{LQR} and c^*. In comparison, the control $u = u_{LQR}(c^*)$ applied to $c = \theta_c$ gives a slightly higher cost (but this may not be the lowest cost for this control as it is likely that another c achieves the minimum) while keeping the maximum cost to 13.4, as compared to a maximum of 17.1 for $u = u_{LQR}(\theta_c)$. Therefore, this example illustrates that by giving up some performance at the nominal plant θ_c, we gain some performance back for other plants in the set.

	$c = \theta_c$	$c = c_{LQR}$	$c = c^*$
$u = u_{LQR}(\theta_c)$	5.6	17.1	16.9
$u = u_{LQR}(c^*)$	7.3	13.3	13.4

Table 1: Cost matrix for different u's and c's.

6 Conclusion

We presented a controller design method for continuous-time plants whose uncertain parameters in the output matrix are known to lie in an ellipsoidal set. This design problem is posed as a minimax problem, where the control and plant uncertainty can be viewed as strategies employed by opposing players in a game, in which the control is chosen to minimize the LQR cost and the plant uncertainty is chosen to maximize it. Without restricting the form of this minimax control, we proved that it is the LQR control for one of the plants in the ellipsoidal set, the worst-case plant. We then proved that this worst-case plant always exists as a fixed point for a certain mapping. A simple heuristic algorithm for computing this fixed point was also given.

References

[1] Eugene L. Allgower and Kurt Georg. *Numerical Continuation Methods.* Springer-Verlag, New York, 1990.

[2] Tamer Basar and Pierre Bernhard. $\mathbf{H}_\infty$-*Optimal Control and Related Minimax Design Problems.* Birkhäuser, Boston, 1991.

[3] G. Belforte, B. Bona, and V. Cerone. Parameter estimation algorithms for a set-membership description of uncertainty. *Automatica*, 26(5):887–898, 1990.

[4] Dennis S. Bernstein and Bong Wie. *Benchmark Problems for Robust Control Design – Proposal for an Invited Session for the 1991 American Control Conference.* October 1991.

[5] Dimitri P. Bertsekas and Ian B. Rhodes. Recursive state estimation for a set-membership description of uncertainty. *IEEE Trans. Aut. Control*, 16(2):117–128, April 1971.

[6] Arthur E. Bryson, Jr. and Yu-Chi Ho. *Applied Optimal Control: Optimization, Estimation, and Control.* Hemisphere Publishing Corporation, Washington, DC, 1975.

[7] Hervé Chapellat and S. P. Bhattacharyya. A generalization of Kharitonov's theorem: Robust stability of interval plants. *IEEE Trans. Aut. Control*, 34(3):306–311, March 1989.

[8] John C. Doyle, Keith Glover, Pramod P. Khargonekar, and Bruce A. Francis. State-space solutions to standard $\mathbf{H}_2$ and $\mathbf{H}_\infty$ control problems. *IEEE Trans. Aut. Control*, 34(8):831–847, August 1989.

[9] Michael K. H. Fan, André L. Tits, and John C. Doyle. Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. *IEEE Trans. Aut. Control*, 36(1):25–38, January 1991.

[10] Joel N. Franklin. *Matrix Theory.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1968.

[11] P. Gahinet and A. J. Laub. Computable bounds for the sensitivity of the algebraic Riccati equation. *SIAM J. Control and Opt.*, 28(6):1461–1480, November 1990.

[12] C. V. Hollot and F. Yang. Robust stabilization of interval plants using lead or lag compensators. In *Proc. IEEE Conf. on Decision and Control*, pages 43–45, Tampa, FL, December 1989.

[13] Samuel Karlin. *Mathematical methods and theory in games, programming, and economics*, volume 2. Addison-Wesley, Reading, MA, 1959.

[14] Robert L. Kosut, Ming Lau, and Stephen Boyd. Identification of systems with parametric and nonparametric uncertainty. In *Proc. American Control Conf.*, pages 2412–2417, San Diego, CA, June 1990.

[15] Ming K. Lau, Stephen Boyd, Robert L. Kosut, and Gene F. Franklin. A robust control design for FIR plants with parameter set uncertainty. In *Proc. American Control Conf.*, pages 83–88, Boston, MA, June 1991.

[16] Ming K. Lau, Robert L. Kosut, and Stephen Boyd. Parameter set estimation of systems with uncertain nonparametric dynamics and disturbances. In *Proc. IEEE Conf. on Decision and Control*, pages 3162–3167, Honolulu, HI, December 1990.

[17] M Milanese and A. Vicino. Estimation theory for nonlinear models and set membership uncertainty. *Automatica*, 27(2):403–408, 1991.

[18] Raymond A. Mills and Arthur E. Bryson, Jr. Minimax state feedback design with norms on initial conditions, disturbances, and parameter changes. In *Proc. American Control Conf.*, pages 1917–1918, Boston, MA, June 1991.

[19] J. P. Norton. *An Introduction to Identification.* Academic Press Inc., 24/28 Oval Road, London NW 1, 1986.

[20] Ihnseok Rhee and Jason L. Speyer. A game theoretic approach to a finite-time disturbance attenuation problem. *IEEE Trans. Aut. Control*, 36(9):1021–1032, September 1991.

[21] Hans Sagan. *Advanced Calculus.* Houghton Mifflin Company, Boston, 1974.

[22] Fred C. Schweppe. Recursive state estimation: Unknown but bounded errors and system inputs. *IEEE Trans. Aut. Control*, 13(1):22–28, February 1968.

[23] Dragoslav D. Siljak. Parameter space methods for robust control design: A guided tour. *IEEE Trans. Aut. Control*, 34(7):674–688, July 1989.

[24] A. J. J. Talman and G. Van Der Laan. *Variable Dimension Fixed Point Algorithms and Triangulations.* Mathematisch Centrum, Amsterdam, 1980.

[25] Bo Wahlberg. System identification using Laguerre models. *IEEE Trans. Aut. Control*, 36(5):551–562, May 1991.

[26] Richard C. Younce and Charles E. Rohrs. Identification with non-parametric uncertainty. In *Proc. IEEE Conf. on Decision and Control*, pages 3154–3161, Honolulu, HI, December 1990.