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A Robust Control Design for FIR Plants with Parameter Set Uncertainty

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Abstract This paper proposes a method of computing the finite-horizon control inputs for FIR plants whose parameters are only known to lie in a set. The parameter set is assumed to be described by an ellipsoidal bound, which could be provided by some identification scheme with a parameter set estimator. The finite-horizon control obtained minimizes the maximum LQR cost from all plants with parameters in the given set. The computation of this robust control is shown to be a convex optimization problem, thus global minimization is guaranteed and many efficient methods are available to compute the minimizing control. In addition, the method can also be used to compute the control for the dual problem in which the plant parameters are known but the initial states of the plant are assumed to lie in a set.

1 Introduction

A problem of great interest in control theory is the design of a controller which can guarantee some level of performance in the presence of plant parameter uncertainty. Kharitonov's theorem provides a necessary and sufficient analysis test for determining the robust stability of polynomials with perturbed coefficients, however, there are few results that exploit Kharitonov's theorem for synthesizing robust controllers, *e.g.*, [4] and [10]. Another approach to this problem is to define a set of nominal values of the uncertain parameters and consider deviations from these nominal values. A comprehensive survey of the different parameter space methods, as opposed to frequency domain methods, can be found in [13].

Motivated by recent work from [11], [12], and [1], where the identified plant parameters are described by ellipsoidal sets, we pose the following problem: given that the plant parameters are known to lie in an ellipsoid, find the finite-horizon control which minimizes the maximum LQR cost from all plants with parameters in the given set. At time k, this minimization produces the control vector $[u(k) \;\; u(k+1) \;\; \cdots \;\; u(k+N)]$, but only $u(k)$ is applied. At time $k+1$, a new minimization problem is solved. This approach of control application is the same as the generalized predictive control described in [5] and [2].

In this paper, we choose to work with finite impulse response (FIR) models for the plant with the assumption that they are accurate models provided they of sufficient lengths. (In doing so, we have also assumed that the plant is stable.) Our goals are to show that the above minimization problem is a convex optimization problem and to design an algorithm to compute the minimizing control. In order to solve the minimization problem, a constrained maximization problem must also be solved. The procedures of which are given in the Appendix. We will also show that the same algorithm can be used to compute the control for the dual problem in which the plant parameters are known but the initial states of the plant are assumed to lie in a set. The paper is organized as follows, after stating the problem in the next section, we will show convexity in section 3 and outline the algorithm. The dual problem of uncertain initial states is considered in section 4. A numerical example is given in section 5. Some concluding remarks are given in section 6.

2 Problem Statement

We shall consider a discrete FIR plant

$$\begin{aligned} y(k) &= b_1 u(k-1) + \cdots + b_m u(k-m) \\ &= \theta^T \phi(k) \end{aligned} \tag{1}$$

where $y(k)$ and $u(k)$ are the output and control of the plant at time k, respectively, and

$$\begin{aligned} \theta &= \left[\; b_1 \;\; b_2 \;\; \cdots \;\; b_m \; \right]^T \\ \phi(k) &= \left[\; u(k-1) \;\; u(k-2) \;\; \cdots \;\; u(k-m) \; \right]^T \end{aligned}$$

The parameter vector of the plant, θ, is assumed to be in a set,

$$\theta \in \Theta \triangleq \{\theta : (\theta - \theta_c)^T \Gamma (\theta - \theta_c) \le 1\} \tag{2}$$

where $\Gamma = \Gamma^T > 0$. Note that Θ describes an ellipsoid in the parameter space with its center at θ_c. The matrix Γ gives the size and orientation of the ellipsoid, *i.e.*, the

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square roots of the reciprocals of the eigenvalues of Γ are the lengths of the semi-axes of the ellipsoid and the eigenvectors of Γ are the directions of the semi-axes.

The plant in (1) can also be represented in state space format,

$$
\begin{aligned}
x(k+1) &= Ax(k) + bu(k) && (3)\\
y(k) &= cx(k) && (4)
\end{aligned}
$$

where

$$
A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},
$$

and

$$
c = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix} = \theta^T
$$

Thus, the states of the FIR plant are

$$
x(k) = \begin{bmatrix} u(k-1) & u(k-2) & \cdots & u(k-m) \end{bmatrix}^T = \phi(k)
$$

Due to past disturbances, the states at some time k_0 are displaced to $\phi(k_0) = \phi_0 \neq 0$, so $y(k_0) \neq 0$. Without loss of generality, we let $k_0 = 0$. We now define the control and output vectors

$$
\begin{aligned}
u &\triangleq \begin{bmatrix} u(0) & u(1) & u(2) & \cdots & u(N) \end{bmatrix}^T \\
y &\triangleq \begin{bmatrix} y(0) & y(1) & y(2) & \cdots & y(N) \end{bmatrix}^T
\end{aligned}
$$

and the quadratic cost function

$$
J_o \triangleq \rho u^T u + y^T y \qquad (5)
$$

where ρ is a weight to trade control effort for regulation. The problem is to find a control which minimizes the cost function for the worst possible plant in Θ, *i.e.*,

$$
u^* = \arg\min_u \left(\max_{\theta \in \Theta} J_o \right) \qquad (6)
$$

Thus, u^* is designed to be robust with respect to the parameter set uncertainty given in (2). Note that if there were no parameter uncertainty in the plant, $\theta = \theta_c$, then (6) becomes

$$
u_{LQR} = \arg\min_u J_o \qquad (7)
$$

which is the standard finite-horizon linear quadratic regulator problem. The optimal control in (7) requires the solution of the discrete Riccati equation, which can be found in texts such as [7, 2].

3 Robust Control Design

We will solve the minimax problem of (6) by showing that it is a convex optimization problem. Note that since $u^T u$ is not a function of θ, we have

$$
u^* = \arg\min_u [J_1(u) + J_2(u)]
$$

where

$$
\begin{aligned}
J_1(u) &= \rho u^T u && (8)\\
J_2(u) &= \max_{\theta \in \Theta} y^T y && (9)
\end{aligned}
$$

We can express y as

$$
y = U\theta
$$

where

$$
U = \begin{bmatrix} u(-1) & u(-2) & \cdots & u(-m) \\ u(0) & u(-1) & \cdots & u(-m+1) \\ u(1) & \ddots & \ddots & \vdots \\ \vdots & & & \vdots \\ u(N-1) & u(N-2) & \cdots & u(N-m) \end{bmatrix}
$$

We now state and prove the following corollary, which states that the maximizer of (9) always lies on the boundary of Θ.

Corollary 1 *Let $\|\cdot\|_2$ denote the Euclidean norm,* i.e.

$$
\|x\|_2^2 \triangleq x^T x
$$

For a fixed matrix U,

$$
f(\theta) = \|U\theta\|_2^2
$$

is convex in θ and

$$
\max_{\theta \in \Theta} \|U\theta\|_2^2 = \max_{\theta \in \Theta_b} \|U\theta\|_2^2 \qquad (10)
$$

where

$$
\Theta_b = \left\{ \theta : (\theta - \theta_c)^T \Gamma (\theta - \theta_c) = 1 \right\} \qquad (11)
$$

Proof of Corollary 1 Let $\alpha \in [0, 1]$, then

$$
\begin{aligned}
& f(\alpha\theta_1 + (1-\alpha)\theta_2) - \alpha f(\theta_1) - (1-\alpha) f(\theta_2) \\
= \; & \|U(\alpha\theta_1 + (1-\alpha)\theta_2)\|_2^2 - \alpha \|U\theta_1\|_2^2 - (1-\alpha)\|U\theta_2\|_2^2 \\
= \; & -\alpha(1-\alpha) \|U(\theta_1 - \theta_2)\|_2^2 \\
\leq \; & 0
\end{aligned}
$$

Thus, $f(\theta)$ is convex in θ. Now let $\theta_1, \theta_2 \in \Theta_b$, then

$$
f(\alpha\theta_1 + (1-\alpha)\theta_2) \leq \alpha f(\theta_1) + (1-\alpha) f(\theta_2)
$$

Since the graph of $f(\theta)$ along the line segment joining any θ_1 and θ_2 lies on or below the line segment with its ends at $f(\theta_1)$ and $f(\theta_2)$, (10) follows. (A different proof of the maximum occurring on the boundary can be found in [14].) □

Thus, the maximizer of (9) is given by

$$\theta^* = \arg\max_{\theta\in\Theta_b} \|U\theta\|_2^2 \qquad (12)$$

Theorem 1 *The functional*

$$J(u) = J_1(u) + J_2(u) \qquad (13)$$

is convex in u.

Proof of Theorem 1 We express y as

$$y = B_1(\theta)u + B_2(\theta)\phi_o \qquad (14)$$

where

$$B_1(\theta) = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ b_1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ b_2 & b_1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ b_m & b_{m-1} & \cdots & b_1 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & b_m & b_{m-1} & \cdots & b_1 & 0 \end{bmatrix}$$

$$B_2(\theta) = \begin{bmatrix} b_1 & b_2 & \cdots & b_{m-1} & b_m \\ b_2 & b_3 & \cdots & b_m & 0 \\ b_3 & \cdots & b_m & 0 & 0 \\ \vdots & b_m & & & \vdots \\ b_m & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

and

$$\phi_o = [\ u(-1) \quad u(-2) \quad \cdots \quad u(-m)\ ]^T$$

Then

$$y^T y = \phi_o^T B_2^T B_2 \phi_o + 2\phi_o^T B_2^T B_1 u + u^T B_1^T B_1 u \qquad (15)$$

The first term on the right-hand side of (15) is constant in u, the second term is linear in u, and the third term $u^T B_1^T B_1 u = \|B_1 u\|_2^2$ is convex in u by Corollary 1. Thus, $y^T y$ is convex in u for each $\theta \in \Theta$. Since the maximum of a set of convex functionals is also convex [3, page 131], $J_2(u)$ is convex. By Corollary 1, $J_1(u) = \rho\|u\|_2^2$ is convex also. Since the sum of convex functionals is convex [3, page 131], $J(u)$ is convex in u. □

With Theorem 1, we are guaranteed that there is a global minimum solution for u^* and many efficient methods are available to compute it. However, we want to point out that although $J_2(u)$ is convex in u, it is not differentiable for all u. We will illustrate this point with the following simple example. Consider the case where $m = 2$ and $N = 1$, so $\theta = [b_1 \ \ b_2]^T$ and $u = [u(0) \ \ u(1)]^T$. Since $y(1)$ does not depend on $u(1)$, we have $u(1) = 0$ and can consider $u = u(0)$. Let Θ be the set of points which lie on the line segment from $\theta_1 = [0.5 \ \ -1]^T$ to $\theta_2 = [1 \ \ 1]^T$, and $\phi_o = [-1 \ \ 1]^T$. As shown in Corollary 1, for a given u, the maximum of $y^T y$ must be at either endpoints of Θ,

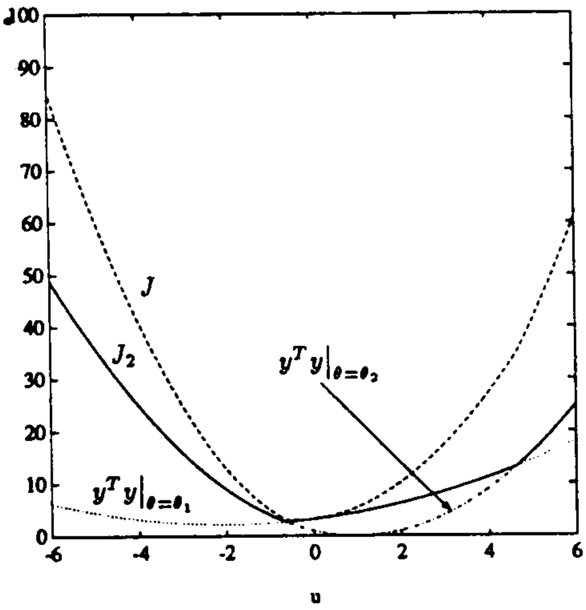

Figure 1: J and J_2 as functions of u.

$$J_2(u) = \max\left(y^T y\big|_{\theta=\theta_1},\ y^T y\big|_{\theta=\theta_2} \right)$$

Figure 1 shows that for this example, there are two points where $J_2(u)$ is not differentiable. Also shown in Figure 1 is $J(u)$ with its minimum at $u^* = -0.4$.

Since $J_2(u)$ is not differentiable for all u, we choose not to use the usual descent methods to find u^*. Instead, we will show that we can easily compute a subgradient of $J(u)$ and apply the ellipsoid algorithm described in [3, pages 324–332].

We first give the definition of a subgradient. If $J : \mathbb{R}^{N+1} \to \mathbb{R}$ is convex, but not necessarily differentiable, then $g \in \mathbb{R}^{N+1}$ is a subgradient of J at u_o if

$$J(u) \geq J(u_o) + g^T(u - u_o) \quad \text{for all } u$$

The set of all subgradients of J at u_o is denoted by $\partial J(u_o)$, the subdifferential of J at u_o. The following two facts from [3, page 300] will be used.

1. Since $J_1(u)$ and $J_2(u)$ are convex, any subgradient of the form $g = g_1 + g_2$ is in $\partial J(u)$, where $g_1 \in \partial J_1(u)$ and $g_2 \in \partial J_2(u)$.
2. Let $y^T y$ from (15) evaluated at θ^* from (12) be de-

noted by

$$J_2(u,\theta^*) = \phi_o^T B_2^T(\theta^*)B_2(\theta^*)\phi_o + 2\phi_o^T B_2^T(\theta^*)B_1(\theta^*)u + u^T B_1^T(\theta^*)B_1(\theta^*)u \quad (16)$$

Since $y^T y$ is convex in u for each $\theta \in \Theta_b$, $g_2 \in \partial J_2(u,\theta^*)$ implies $g_2 \in \partial J_2(u)$. In the event that there are more than one maximum, we only need to pick one.

Thus, from (8) and (16) the subgradient of J at u is given by

$$g = 2\rho u + \left[2B_1^T(\theta^*)B_2(\theta^*)\phi_o + 2B_1^T(\theta^*)B_1(\theta^*)u\right] \quad (17)$$

The computation of θ^* is not difficult, but the derivation is rather long. To avoid breaking the flow of this section, the method of finding θ^* is given in the Appendix. The ellipsoid algorithm for computing $u^* \in \mathbb{R}^K$ is as follows:

1. Select any u_1 and E_1 such that u^* is in the initial ellipsoid,
$$u^* \in \left\{u : (u-u_1)^T E_1^{-1}(u-u_1)\right\}$$
2. $k \leftarrow 0$;
3. $k \leftarrow k+1$;
4. Compute any $g_k \in \partial J(u_k)$:
   (a) Compute z^* from Theorem 2;
   (b) Compute θ^* from (31);
   (c) Compute g_k from (17);
5. Compute new ellipsoid:
$$\begin{aligned}\tilde{g} &\leftarrow \frac{g_k}{\sqrt{g_k^T E_k g_k}}\\ u_{k+1} &\leftarrow u_k - \frac{E_k\tilde{g}}{K+1}\\ E_{k+1} &\leftarrow \frac{K^2}{K^2-1}\left(E_k - \frac{2}{K+1}E_k\tilde{g}\tilde{g}^T E_k\right)\end{aligned}$$
6. If $\sqrt{g_k^T E_k g_k} > \epsilon$, go to step 3.

The stopping criterion in step 6 guarantees that on exit, $J(u_k)$ is within ϵ of $J(u^*)$.

4 Uncertain Initial States

In this section, we will consider the dual problem in which the parameter vector θ of the plant is known, but the initial states of the plant ϕ_o is assumed to be in a set similar to (2),

$$\phi_o \in \Phi \triangleq \{\phi_b : (\phi_o - \phi_c)^T \Gamma_\phi (\phi_o - \phi_c) \le 1\} \quad (18)$$

The problem posed in (6) now becomes

$$\begin{aligned}u^* &= \arg\min \max_{\phi_o\in\Phi} J_o \quad (19)\\ &= \arg\min \left[\rho u^T u + \max_{\phi_o\in\Phi} y^T y\right]\end{aligned}$$

Note that $y^T y$ from (15) is convex in ϕ_o for a given u. This means that the maximum of $y^T y$ lies on the boundary of Φ, Φ_b. Furthermore, using the same arguments from the proof of Theorem 1,

$$J_\phi(u) = \rho u^T u + \max_{\phi_o\in\Phi_b} y^T y$$

can be shown to be convex in u. Therefore, all we need to show is that we can compute a subgradient of $J_\phi(u)$,

$$g_\phi = 2\rho u + 2B_1^T B_1 u + 2B_1^T B_2 \phi_o^* \quad (20)$$

where

$$\phi_o^* = \arg\max_{\phi_o\in\Phi_b} y^T y$$

From (14), we have

$$\phi_o^* = \arg\max_{\phi_o\in\Phi_b} \|B_2\phi_o + B_1 u\|_2$$

This is similar to the form of (12) except that we have the extra term $B_1 u$. Thus, if we solve for θ^* with

$$q = -(B_2\phi_c + B_1 u)$$

in (29) and replace Γ and θ_c of (2) with Γ_ϕ and ϕ_c from (18), we have

$$\phi_o^* = \theta^*$$

Therefore, u^* in (19) can be computed by the same ellipsoid algorithm given in Section 4, where the subgradient is now computed using (20).

5 Numerical Example

For our example, we use a 10-tap FIR plant, *i.e.*, $m = 10$. The control vector u has $N = 10$, so if $u \equiv 0$, the output will be zero after 10 delays, $y(10) = 0$. The parameter ellipsoid Θ in (2) is a 10-dimensional ball with a radius of 5 and center at θ_c. θ_c, plotted in Figure 2 with the '+' symbol, is the first ten terms of the impulse response from the transfer function

$$\frac{10z(z+0.7\cos(\pi/4))}{z^2 - 2(0.7)\cos(\pi/4)z + 0.7^2}$$

The initial state of the plant,

$$x(0) = [u(-1)\ \ u(-2)\ \ \cdots\ \ u(-10)]^T$$

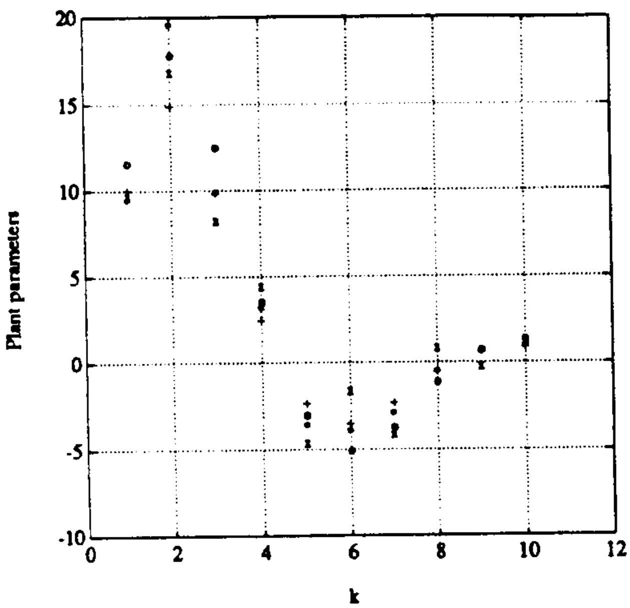

Figure 2: Plant parameters: o – θ_1, x – θ_2, * – θ_3, and + – θ_c.

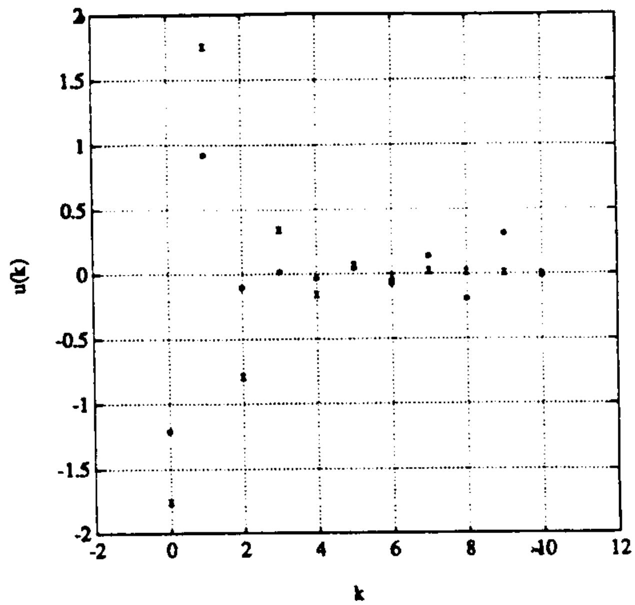

Figure 3: Controls: x – u_{LQR} and * – u^*.

is scaled such that $\|x(0)\|_2 = 1$.

Using $\rho = 1$, we will compare the cost J in (13) associated with three controls, $u_1 = 0$, $u_2 = u_{LQR}$, and $u_3 = u^*$, where u_{LQR} is given by (7) with $\theta = \theta_c$. The controls u_{LQR} and u^* are plotted in Figure (3), where $\|u_{LQR}\|_2 = 2.63$ and $\|u^*\|_2 = 1.58$. We now define three plants from Θ,

$$\theta_i \triangleq \arg\max_{\theta\in\Theta} (J_o|_{u=u_i}) \qquad i = 1,2,3$$

They are the worst-case plants for their associated controls and are plotted in Figure 2. Table 1 shows the cost matrix, C, for the different plants and controls. We make the following observations from C:

1. For $i = 1, 2, 3$, $C(i,i)$ is the largest in each row, as the θ_i's are chosen that way.
2. u_{LQR} has the lowest cost for θ_c, 403, but only 8% lower than u^*.
3. u^* has the lowest maximum cost, 697, 48% lower than the maximum cost from u_{LQR} and 87% lower than that from $u = 0$. Thus, the robust design performed as expected.

	θ_1	θ_2	θ_3	θ_c
$u_1 = 0$	1306	902	1136	837
$u_2 = u_{LQR}$	587	1031	785	403
$u_3 = u^*$	655	632	697	437

Table 1: Cost matrix for different u's and θ's.

6 Concluding Remarks

We have shown in this paper that given that the FIR plant parameters are known to lie in an ellipsoid, finding the finite-horizon control to minimize the maximum LQR cost from all plants with parameters in the given set is a convex optimization problem. An algorithm is given to compute this minimizing control. Although the algorithm can also compute the minimizing control when the plant parameters are known but the initial states of the plant are in an ellipsoid, it would be desirable to minimize the maximum over both parameter and initial state uncertainties simultaneously. Furthermore, we would like to extend our method to the infinite-horizon case for infinite impulse response (IIR) plants. These are areas of our current research.

7 Appendix

Given the following matrices,

$$U \in \mathbb{R}^{(N+1)\times m} \quad (21)$$
$$\Gamma \in \mathbb{R}^{m\times m},\ \Gamma = \Gamma^T > 0 \quad (22)$$
$$\theta, \theta_c \in \mathbb{R}^m \quad (23)$$

we want to find the maximizer θ^* in (12). This is similar to the least squares problem with quadratic and linear constraints, which was investigated in [8] and [9]. However, we are seeking a maximizer as compared to a minimizer.

Since Γ is symmetric, we can diagonalize it by a unitary matrix,

$$\Gamma = T\Lambda T^T$$

where Λ is diagonal with eigenvalues of Γ and the columns of T are eigenvectors of Γ. We now transform Θ_b in (11) to the unit ball,

$$B = \left\{z : z^T z = 1\right\} \quad (24)$$

where

$$z = \Lambda^{\frac{1}{2}} T^T(\theta - \theta_c) \quad (25)$$

Substituting

$$\theta = T\Lambda^{-\frac{1}{2}} z + \theta_c \quad (26)$$

into (12), we have

$$z^* = \arg \max_{z^T z = 1} \|Dz - q\|_2^2 \quad (27)$$

where

$$D = UT\Lambda^{-\frac{1}{2}} \quad (28)$$

$$q = -U\theta_c \quad (29)$$

Define

$$\Omega \triangleq D^T D$$

$$\beta \triangleq D^T q$$

then

$$z^* = \arg \max_{z^T z = 1} z^T \Omega z - 2\beta^T z \quad (30)$$

Substituting z^* into (26), θ^* in (12) is given by

$$\theta^* = T\Lambda^{-\frac{1}{2}} z^* + \theta_c \quad (31)$$

To find z^* in (30), we introduce the Lagrange multiplier λ and adjoin the constraint, $z^T z = 1$,

$$L = z^T \Omega z - 2\beta^T z + \lambda\left(1 - z^T z\right)$$

Necessary conditions for the stationary points are

$$\frac{\partial L}{\partial z} = 2\Omega z - 2\beta - 2\lambda z = 0$$

$$\frac{\partial L}{\partial \lambda} = 1 - z^T z = 0$$

or

$$\Omega z = \lambda z + \beta \quad (32)$$

$$z^T z = 1 \quad (33)$$

The problem of finding all the stationary points of such a second-degree polynomial on the unit sphere was first investigated in [6], but the computation of the solution was not considered there. A proof similar to the one given in [8], however, can be used to show the following:

Corollary 2 *If (z_1, λ_1) and (z_2, λ_2) satisfy (32) and (33) and $\lambda_1 > \lambda_2$, then*

$$z_1^T \Omega z_1 - 2\beta^T z_1 > z_2^T \Omega z_2 - 2\beta^T z_2 \quad (34)$$

Thus, in place of the maximization problem in (30), we need to solve the Lagrange equations (32) and (33) with

$$\lambda = maximum \quad (35)$$

In [9], it was shown that (32) and (33) can be transformed to a quadratic eigenvalue problem,

$$(\Omega - \lambda I)^2 \eta = \beta\beta^T \eta$$

Furthermore, the quadratic eigenvalue problem can be reduced to an ordinary eigenvalue problem by finding the eigenvalues of

$$M = \begin{bmatrix} \Omega & -I \\ -\beta\beta^T & \Omega \end{bmatrix}$$

The solution of (30) is summarized in the following theorem:

Theorem 2 *Let λ^* be the largest eigenvalue of M, then there are two possible cases for the maximizer of (30):*

1. *If λ^* is not an eigenvalue of Ω, then $z^* = (\Omega - \lambda^* I)^{-1}\beta$.*
2. *If λ^* is an eigenvalue of Ω, then let $\nu = (\Omega - \lambda^* I)^\dagger \beta$, where $\dagger$ denotes the pseudoinverse, and*
   - *(a) If $z = \nu$ satisfies (32) and (33), then $z^* = \nu$.*
   - *(b) If $z = \nu$ satisfies (32) and $\nu^T \nu < 1$, then $z^* = \nu + \zeta$ is one of many solutions, where ζ is an eigenvector to the eigenvalue λ^* of Ω with $\zeta^T \zeta = 1 - \nu^T \nu$.*

Proof of Theorem 2 In [9], the minimization of (30) was analyzed. Due to Corollary 2, we can apply all the results from [9] by replacing the smallest eigenvalue of M with the largest eigenvalue. □

References

[1] G. Belforte, B. Bona, and V. Cerone. Parameter estimation algorithms for a set-membership description of uncertainty. *Automatica*, 26(5):887–898, 1990.

[2] Robert R. Bitmead, Michel Gevers, and Vincent Wertz. *Adaptive Optimal Control: The Thinking Man's GPC.* Prentice Hall of Australia Pty Ltd, 1990.

[3] Stephen P. Boyd and Craig H. Barratt. *Linear Controller Design - Limits of Performance.* Prentice Hall, Englewood Cliffs, NJ, 1991.

[4] H. Chapellat and S. P. Bhattacharyya. A generalization of kharitonov's theorem: Robust stability of interval plants. *IEEE Trans. Aut. Control*, 34:306–311, 1989.

[5] D. W. Clarke, C. Mohtadi, and P. S. Tuffs. Generalized predictive control: Parts I and II. *Automatica*, 23:137–160, 1987.

[6] George E. Forsythe and Gene H. Golub. On the stationary values of a second-degree ploynomial on the unit sphere. *J. Soc. Indust. Appl. Math.*, 13(4):1050–1068, Dec 1965.

[7] Gene F. Franklin, J. David Powell, and Michael L. Workman. *Digital Control of Dynamic Systems.* Addison-Wesley Publishing Company, Inc, 1990.

[8] Walter Gander. Least squares with a quadratic constraint. *Numerische Mathematik*, 36:291–307, 1981.

[9] Walter Gander, Gene H. Golub, and Urs von Matt. A constrained eigenvalue problem. *Linear Algebra and Its Applications*, pages 815–839, 1989.

[10] C. V. Hollot and F. Yang. Robust stabilization of interval plants using lead or lag compensators. In *Proc. IEEE Conf. on Decision and Control*, pages 43–45, Tampa, FL, Dec 1989.

[11] Robert L. Kosut, Ming Lau, and Stephen Boyd. Identification of systems with parametric and nonparametric uncertainty. In *Proc. American Control Conf.*, pages 2412-2417, San Diego, CA, June 1990.

[12] Ming K. Lau, Robert L. Kosut, and Stephen Boyd. Parameter set estimation of systems with uncertain nonparametric dynamics and disturbances. In *Proc. IEEE Conf. on Decision and Control*, pages 3162–3167, Honolulu, HI, Dec 1990.

[13] Dragoslav D. Siljak. Parameter space methods for robust control design: A guided tour. *IEEE Trans. Aut. Control*, 34(7):674–688, July 1989.

[14] Emil Spjøtvoll. A note on a theorem of Forsythe and Golub. *SIAM J. Appl. Math.*, 23(3):307–311, Nov 1972.