# Convergence of Departures in Tandem Networks of ·/GI/∞ Queues

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We consider an infinite series of independent and identical ./GI/infinity queues fed by an arbitrary stationary and ergodic arrival process. Using couplings for random walks, we show that the limiting distribution of the departure process from the nth queue converges in distribution either to a Poisson process or to a stationary v Poisson process (defined below) depending on the joint distribution of the original arrival process and the service process.



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# Convergence of Departures in Tandem Networks of $\cdot/\mathrm{GI}/\infty$ Queues


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### Abstract

We consider an infinite series of independent and identical $\cdot/\mathrm{GI}/\infty$ queues fed by an arbitrary stationary and ergodic arrival process, $\mathrm{A}^1$. Let $\mathrm{A}^i$ be the arrival process to the $i^{th}$ node and let $\nu^i$ be the law of $\mathrm{A}^i$. Denote by $\mathcal{T}(\cdot)$ the input-output map of the $\cdot/\mathrm{GI}/\infty$ node; that is, $\nu^{i+1} = \mathcal{T}(\nu^i)$. It is known that the Poisson process is a fixed point for $\mathcal{T}$. In this paper, we are interested in the asymptotic distribution of the departure process from the $n^{th}$ node, $\nu^{n+1} = \mathcal{T}^n(\nu^1)$, as $n \to \infty$. Using couplings for random walks, this limiting distribution is shown to be either a Poisson process or a stationary $\nu$-Poisson process (defined below) depending on the joint distribution of $\mathrm{A}^1$ and the service process. This generalizes a result of Vere-Jones [11] and is similar in flavour to [10] where Poisson convergence is established for departures from a series of exponential server queues using coupling methods.


## 1 Introduction

Infinite server queues which dispense i.i.d. but otherwise arbitrarily distributed service times (symbolically, $\cdot/\mathrm{GI}/\infty$ queues) are well-known examples of quasi-reversible nodes. Their behaviour under Poisson process inputs is well-understood; in particular, they are known to possess the Poisson-in-Poisson-out property (see, for example, [8]). In this paper couplings for random walks are used to study the asymptotic behaviour of departure processes in large tandem networks of $\cdot/\mathrm{GI}/\infty$ queues under non-Poisson process inputs. Vere-Jones [11] used the method of probability generating functionals to study this model. He showed that as a stationary, ergodic and *weakly mixing* point process passes through a series of independent and identical $\cdot/\mathrm{GI}/\infty$ nodes, it converges to the Poisson process in distribution (we assume throughout that the service times of the $\cdot/\mathrm{GI}/\infty$ nodes are not equal to a constant almost surely; if this is the case, the $\cdot/\mathrm{GI}/\infty$ queue is a pure delay system).

Simple counterexamples show that this result can be false if the weak mixing condition is dropped, i.e. the successive departure processes can converge to a non-Poisson process limit. For example, consider the deterministic arrival process with inter-arrival times exactly equal to one unit passing through a series of independent and identical $\cdot/\mathrm{GI}/\infty$ queues with integer-valued service times. Since the points of all the successive departure processes will be separated by integer distances, Poisson convergence is impossible. However, as we shall see, the departure processes *do converge* in this case also (albeit to a non-Poisson limit). The coupling methods used in this paper allow one to obtain a complete characterization of this limiting process in a natural way. The details of how this is done are given in Section 2.2.

In addition to [11], other studies of queueing tandems may be found in [2], [3], [6] and [10]. In [2] Bambos and Prabhakar derive the asymptotic completion times of a finite number of jobs flowing through a long series of queues where the service time of each job at different queues forms a stationary and ergodic sequence. A similar model is considered by Glynn and Whitt [6] with the service times of individual jobs at the various nodes being distributed in an i.i.d. fashion. Bambos and Walrand [3] consider the G/G/1 queue as an operator that maps interarrival time sequences into interdeparture time sequences and study the limiting behaviour of flows in a series of $\cdot/\mathrm{G}/1$ nodes. Mountford and Prabhakar [10] show that the limit from passing a stationary and ergodic process of rate $\alpha$ through an infinite series of independent $\cdot/\mathrm{M}/1$ queues of service rate 1 is a Poisson process. [10] is similar in flavour to the present work in that both establish Poisson convergence results using coupling methods. We conclude the introduction by providing the following brief summary of the rest of the paper.

Random walk couplings are used to show that when *any* stationary and ergodic arrival process is passed through a sequence of $\cdot/\mathrm{GI}/\infty$ queues, the successive departure processes converge in distribution either to a Poisson process or to a stationary $\nu$-Poisson process (defined later) depending on the type of service distribution. In particular, our result includes Vere-Jones' result as a special case. Our methods also allow easy extensions to the case of departures from a series of non-identically distributed queues.

## 2 Convergence of Departures in $\cdot/\mathrm{GI}/\infty$ Queueing Tandems

In this section we study departures from an infinite series of independent and identical $\cdot/\mathrm{GI}/\infty$ nodes. Suppose customers arrive at the first node in this series according to a process $\mathrm{A}^1$ given by

$$\mathrm{A}^1 = \sum_{n=-\infty}^{\infty} \delta_{t_n^a}, \tag{1}$$

where $\cdots \leq t_{-1}^a \leq t_0^a \leq 0 < t_1^a \leq \cdots \leq t_n^a \leq t_{n+1}^a \leq \cdots$ pathwise and $\delta_x$ denotes the point mass at $x$. The random variable $t_n^a$ specifies the arrival time of the $n^{th}$ customer of $\mathrm{A}^1$. Note that we allow non-simple point process (i.e. batch process) inputs. We will suppose throughout that arrival processes are stationary and ergodic with respect to the transformations $\{\Theta_t, t \in \mathcal{R}\}$, where $\Theta_t \circ \mathrm{A}^1 = \sum_{n=-\infty}^{\infty} \delta_{t_n^a - t}$. Suppose that $\alpha = E(N^{A^1}(0,1])$ is the rate of $\mathrm{A}^1$, where

$$N^{A^1}(0,1] = \sum_{n=-\infty}^{\infty} 1_{\{0 < t_n^a \leq 1\}}$$

is the number of points of $\mathrm{A}^1$ in $(0,1]$.

Let the service time of the $n^{th}$ customer at the $m^{th}$ queue be given by $\sigma_m^n$. Then, the departure process from the $(m-1)^{th}$ queue, $\mathrm{A}^m$, is completely specified by the sequence of departure times

$$\left\{ t_n^a + \sum_{k=1}^{m-1} \sigma_k^n ; n \in \mathcal{Z} \right\}.$$

We are interested in the asymptotic distribution of $\mathrm{A}^m$ as $m \to \infty$.

**Definition 1:** The service time of a customer is said to be *lattice with span c* if the possible values of the service times are integer multiples of $c$. Otherwise the service is said to be *non-lattice.*

Suppose customers $x$ and $y$ arrive at a series of independent and identical $\cdot/\mathrm{GI}/\infty$ queues. If their service times at queue $k$ are $\sigma_k^x$ and $\sigma_k^y$ respectively, then $S_m = \sum_{k=1}^m \sigma_k^x - \sigma_k^y = \sum_{k=1}^m Z_k$ is the difference in their total service times in the first $m$ queues. In the sequel we will often say that the service of a customer is recurrent or transient, when we actually mean to say that the random walk, $S_m$, induced by the service times is recurrent or transient.

Proposition 1, the main result of this section, introduces the key ideas for establishing weak convergence of departures by first considering tandems of identical $\cdot/\mathrm{GI}/\infty$ queues whose service is non-lattice and recurrent. We then show how the same ideas generalize to other cases (lattice and/or transient service) with appropriate modifications.

**Proposition 1:** *The limit from passing a stationary and ergodic arrival process of rate $\alpha$, $\mathrm{A}^1 = \sum_{n=-\infty}^{\infty} \delta_{t_n^a}$, through an infinite series of independent, identical $\cdot/GI/\infty$ queues whose service is non-lattice and recurrent is a Poisson process of rate $\alpha$.*

**Preliminary sketch of proof:** We will couple $\mathrm{A}^1$ with an independent rate $\alpha$ Poisson process $\mathrm{P}^1 = \sum_{n=-\infty}^{\infty} \delta_{t_n^p}$ by matching up points (or customers) of $\mathrm{A}^1$ and $\mathrm{P}^1$. Customers belonging to $\mathrm{A}^1$ are coloured blue and customers belonging to $\mathrm{P}^1$ are coloured red. So long as a red and a blue customer evolve independently of each other, they perform a random walk as regards to their inter-arrival times at each queue in the series. By noting that this random walk is non-lattice and recurrent we conclude that at some finite queue these two customers will be within $\delta$ of each other, where $\delta$ is arbitrarily small. Once they are sufficiently close, we say they are "coupled" (at which point they are coloured yellow) and let them evolve together forever. Thus, eventually all the points of $\mathrm{A}^n$ and $\mathrm{P}^n$ are close to each other and we get the required weak convergence, since the $\mathrm{P}^n$'s are all Poisson. In matching up customers of $\mathrm{A}^1$ and $\mathrm{P}^1$ for the purpose of coupling them, we cannot just match up the $k^{th}$ customers (who arrive at times $t_k^a$ and $t_k^p$) because as $k$ becomes very large, the difference in their arrival times $(t_k^a - t_k^p)$ will become very large - typically of the order of $\sqrt{k}$. To ensure that we are coupling customers that are not separated by more than a fixed distance, say $R$, we divide time into large blocks of length $R$. The choice of $R$ is dictated by our desire to ensure that there are approximately the same number of blue and red customers in the interval $(jR,\ (j+1)R]$, for $j \in \mathcal{Z}$. The details are as follows.

**Proof of Proposition 1:** The desired weak convergence (see Daley and Vere-Jones [4], Chapter 9 for details on the weak convergence of point processes) follows once we show that for each non-negative continuous function $f$ on $\mathcal{R}$ with compact support

$$\int f \, d\mathrm{A}^n \ \xrightarrow{D} \ \int f \, d\mathrm{P},$$

where $\mathrm{P}$ is a Poisson process of rate $\alpha$. Since for every $n$, $\mathrm{P}^n$ is a rate $\alpha$ Poisson process, it will suffice to show that

$$\int f \, d\mathbf{A}^n - \int f \, d\mathbf{P}^n \xrightarrow{pr} 0. \tag{2}$$

Without loss of generality, suppose $[0, N]$ contains the support of $f$.

Fix $\epsilon > 0$ arbitrarily small. We will show that the point processes $\mathbf{A}^n$ and $\mathbf{P}^n$ can be coupled so that for $n$ sufficiently large

$$E\left|\int f \, d\mathbf{A}^n - \int f \, d\mathbf{P}^n\right| \tag{3}$$

is bounded by a multiple of $\epsilon$. Since $f$ has compact support it is uniformly continuous. Hence we can find $\delta < \epsilon$ so that $\epsilon > \sup(|f(x) - f(y)| : |x - y| < \delta)$.

By the ergodic theorem,

$$\frac{1}{N}\left(N^{A^1}(0,\ N],\ N^{P^1}(0,\ N]\right) \xrightarrow{a.s.} (\alpha,\ \alpha) \text{ as } N \to \infty$$

where $N^{A^1}(0,\ N]$ $\left(N^{P^1}(0,\ N]\right)$ is the number of points of $\mathbf{A}^1$ ($\mathbf{P}^1$) in $(0,\ N]$. Hence there is a sufficiently large $R \in \mathcal{R}$ such that the chance that both $N^{A^1}(0,\ R]$ and $N^{P^1}(0,\ R]$ are between $R(\alpha - \frac{\epsilon}{2})$ and $R(\alpha + \frac{\epsilon}{2})$ is greater than $1 - \epsilon$. Fix such an $R$. Let $U$ be a random variable chosen uniformly on $[0, R]$, independently of the two point processes $\mathbf{A}^1$ and $\mathbf{P}^1$ and of the service times. Divide up the time axis into disjoint intervals of length $R$, $(jR + U,\ (j+1)R + U] j \in \mathcal{Z}$. Say that an interval $(jR + U,\ (j+1)R + U]$ is "good" if both $\mathbf{A}^1$ and $\mathbf{P}^1$ have at least $R(\alpha - \frac{\epsilon}{2})$ points in it. By the joint stationarity of $\mathbf{A}^1$ and $\mathbf{P}^1$, the chance that for any $j \in \mathcal{Z}$ such an interval is good is greater than $1 - \epsilon$. Now, we match up customers as follows.

1) If interval $(jR + U,\ (j+1)R + U]$ is not good, then no customers in the interval (belonging to either $\mathbf{A}^1$ or $\mathbf{P}^1$) are matched.

2) If the interval $(jR + U,\ (j+1)R + U]$ is good, then let the customers of $\mathbf{A}^1$ and $\mathbf{P}^1$ in this interval be matched up, on a one-to-one basis from the left end of the interval to the right end of the interval, possibly with a few unmatched customers of $\mathbf{A}^1$ or $\mathbf{P}^1$ (not both) remaining at the right end of the interval. That is, if the arrival times of the customers of $\mathbf{A}^1$ and $\mathbf{P}^1$ in this interval are, respectively, $x_1^j \le x_2^j \le \cdots \le x^j_{\lfloor R(\alpha - \frac{\epsilon}{2})\rfloor} \le \cdots$, and $y_1^j \le y_2^j \le \cdots \le y^j_{\lfloor R(\alpha - \frac{\epsilon}{2})\rfloor} \le \cdots$, then match up the customer of $\mathbf{A}^1$ at $x_i^j$ with the customer of $\mathbf{P}^1$ at $y_i^j$ for $i \le \lfloor R(\alpha - \frac{\epsilon}{2})\rfloor$. Once the customers at $x_i^j$ and $y_i^j$ are matched up, we say they are "partners".

(**Remark:** Note that without the random variable $U$, the distribution of unmatched customers over $(jR,\ (j+1)R]$ will be uneven with more unmatched customers being found at the end of the interval than at the beginning. The addition of $U$ makes the matching operation, and hence the process of unmatched customers, time stationary.)

Now consider two partners that arrive at times $x_i^j$ and $y_i^j$ for some $j \in \mathcal{Z}$. $\tau^1 = x_i^j - y_i^j$ is the difference between their arrival times to the first queue. If $\sigma_k^a$ and $\sigma_k^p$ are their service times at the $k^{th}$ queue, then the random variable

$$\tau^m = \tau^1 + \sum_{i=1}^{m-1} \sigma_i^a - \sigma_i^p = \tau^1 + \sum_{i=1}^{m-1} Z_i \tag{4}$$

is the difference in the arrival times of these customers at the $m^{th}$ queue. Because the service is recurrent and non-lattice, $\tau^m$ is a recurrent, non-lattice random walk on $\mathcal{R}$ with starting value $\tau^1$. Thus $|\tau^m| < \delta$ for some $m(\omega)$ a.s. where $\delta$ was chosen at the start of the proof. By the coupling scheme, $|\tau^1| \leq R$ a.s. and so the distribution of the first such $m(\omega)$ is tight over matched pairs of partners.

Once the two partners of $\mathbf{A}^1$ and $\mathbf{P}^1$, situated initially at times $x_i^j$ and $y_i^j$ respectively are within $\delta$ of each other, they are said to be "coupled" and are coloured yellow. We then let them receive *identical* service times at all subsequent queues. This ensures that they will forever remain within $\delta$ of each other once they are coupled. Similarly all the other partners couple at some queue or the other a.s. and then evolve together. Let $\mathbf{C}_A^n$ and $\mathbf{C}_P^n$ be, respectively, the set of points of $\mathbf{A}^n$ and $\mathbf{P}^n$ that are coupled.

Now

$$\int f\, d\mathbf{A}^n = \int f\, d\mathbf{C}_A^n + \int f\, d(\mathbf{A}^n - \mathbf{C}_A^n) \quad \text{and}$$

$$\int f\, d\mathbf{P}^n = \int f\, d\mathbf{C}_P^n + \int f\, d(\mathbf{P}^n - \mathbf{C}_P^n).$$

Consider the random variable $\int f\, d(\mathbf{A}^n - \mathbf{C}_A^n)$. Now

$$E\left[\left|\int f\, d(\mathbf{A}^n - \mathbf{C}_A^n)\right|\right] \leq f_{max} E(\#\text{ of points of } \mathbf{A}^n - \mathbf{C}_A^n \text{ in } (0, N]). \qquad (5)$$

The points of $\mathbf{A}^n - \mathbf{C}_A^n$ in $[0, N)$ are either those that have partners but have not coupled with (come within $\delta$ of) them yet; or those that did not have partners in the first place, i.e. the unmatched points of $\mathbf{A}^1$.

The expected density of unmatched points of $\mathbf{A}^1$ (and hence of $\mathbf{A}^n$ for each $n$) is less than or equal to $\alpha$ minus the expected density of matched customers. This in turn is less than or equal to $\alpha - (\alpha - \epsilon/2)(1-\epsilon) \leq \epsilon(\alpha + 1/2)$.

For each matched $x \in \mathbf{A}^1$, the probability of not being coupled after the $n^{th}$ queue depends only on $n$ and the relative displacement of $x$ from its partner at the first stage. As this displacement is bounded by $R$, the probability of not being coupled is bounded by some $c(R, n)$ which tends to zero as $n$ tends to infinity for fixed $R$. The events of matched pairs becoming coupled are mutually independent, given their initial positions in $\mathbf{A}^1$ and $\mathbf{P}^1$. Therefore the density of matched but uncoupled points in $\mathbf{A}^n$ is bounded by $\alpha c(R, n)$.

Putting the two paragraphs together we find that the expected number of uncoupled points of $\mathbf{A}^n$ in $[0, N)$ is at most $\epsilon(\alpha + 1/2) + \alpha c(R, n)$. A similar bound holds for the uncoupled points of $\mathbf{P}^n$. Therefore, for $n$ sufficiently large, the chance that there is an uncoupled customer in $[0, N)$ for either $\mathbf{A}^n$ or $\mathbf{P}^n$ is bounded by $3\epsilon(1/2 + \alpha)$. Using this in (5) we get that

$$E\left[\left|\int f\, d(\mathbf{A}^n - \mathbf{C}_A^n)\right|\right] \leq f_{max}\, 3\epsilon(\alpha + 1/2).$$

To establish the theorem, it now suffices to bound $\left|\int f\, d\mathbf{C}_A^n - \int f\, d\mathbf{C}_P^n\right|$.

Divide the points of $\mathbf{C}_A^n$ in the interval $[0, N]$ into $\mathbf{G}_A^n$ and $\mathbf{B}_A^n$, where $\mathbf{G}_A^n$ consists of those ("good") points of $\mathbf{C}_A^n$ whose partners also fall in the interval $[0, N]$. $\mathbf{B}_A^n$ then consists of those ("bad") points of $C_A^n$ whose partners fall in the set $[-\delta, 0) \cup (N, N+\delta]$. Likewise define $G_P^n$ and $B_P^n$. With this,

$$\left|\int f\, dC_A^n - \int f\, dC_P^n\right| \le \left|\int f\, d(G_A^n - G_P^n)\right| + \left|\int f\, dB_A^n\right| + \left|\int f\, dB_P^n\right|.$$

The last two terms in the right-hand-side of the above expression are bounded by $2\alpha\delta f_{max} < 2\alpha\epsilon f_{max}$ in expectation, since $2\alpha\delta$ bounds the expected number of points of $B_A^n$ and $B_P^n$. Thus,

$$\begin{aligned}
&E\left|\int f\, dC_A^n - \int f\, dC_P^n\right| \\
&\le\ E\left|\int f\, d(G_A^n - G_P^n)\right| + 2\alpha\epsilon f_{max} \\
&=\ E\left(\sum_{k=0}^{\infty}\left|\int f\, d(G_A^n - G_P^n)\right| 1_{\{\#\text{ of pts of } G_P^n \text{ in } [0,\, N]=k\}}\right) + 2\alpha\epsilon f_{max} \\
&\le\ \sum_{k=0}^{\infty} k\epsilon\, 1_{\{\#\text{ of pts of } G_P^n \text{ in } [0,\, N]=k\}} + 2\alpha\epsilon f_{max} \\
&\le\ \epsilon\alpha N + 2\alpha\epsilon f_{max},
\end{aligned}$$

where the last inequality is due to the fact that the expected number of points of $G_P^n$ is close to (but below) $\alpha N$.

Thus, $E\left|\int f\, dA^n - \int f\, dP^n\right|$ is bounded by a multiple of $\epsilon$; and since $\epsilon$ is arbitrarily small, we obtain the desired weak convergence. ∎

### 2.1 Taking care of transience

Going back to equation (4), suppose now that the random walk $\tau^m = \tau^1 + \sum_{k=1}^{m-1} \sigma_k^x - \sigma_k^y$ is transient. We will use what is essentially Ornstein's Coupling ([9], [5] page 281) to modify the assignment of service times to partners in such a way that $\tau^m$ becomes recurrent again. We assume here that service times are non-lattice. The service times $\sigma_k^x$ and $\sigma_k^y$ are now chosen according to the following rules.

• For each $k$, either $\sigma_k^x$ and $\sigma_k^y$ are both bigger than $M$ (to be specified later), or they are both smaller than or equal to $M$.

• If they are both bigger than $M$, then let $\sigma_k^x = \sigma_k^y$. If not, choose $\sigma_k^x$ and $\sigma_k^y$ to be less than or equal $M$ *independently* of each other (and of the service times of other customers).

In other words, big jumps are taken together and small jumps are taken independently. With this modification, the random walk $\tau^m = \tau^1 + \sum_{k=1}^{m-1} \sigma_k^x - \sigma_k^y$ then becomes recurrent because $\sigma_k^x - \sigma_k^y$ is symmetric and bounded in absolute value by $M$.

We are now left with having to specify $M$. Even though the service times $\sigma_k^x$ are non-lattice, when restricted to be less than some number $M$, they may be lattice. For example, suppose the possible values of $\sigma_k^x$ are $\{i - \frac{1}{2^{i-1}},\ i \in \mathcal{Z}^+\}$. Then the possible values of $\sigma_k^x$ restricted to be less than or equal to any number $K$ are lattice with span $\frac{1}{2^{K-1}}$. However, by choosing $K$ large enough we can make the span as small as we wish. Thus, in general when given $\sigma_k^{\bar{x}}$, choose $M$ to be large enough that the span of $\sigma_k^{\bar{x}}$ restricted to be less than $M$ is smaller than $\epsilon$. Now the methods of Proposition 1 can be applied directly and we obtain the desired weak convergence. We have thus proved the following theorem.

**Theorem 1:** *The limit from passing a stationary and ergodic arrival process of rate $\alpha$, $\mathbf{A}^1 = \sum_{n=-\infty}^{\infty} \delta_{t_n^a}$, through an infinite series of independent, identical $\cdot/GI/\infty$ queues whose service is non-lattice is a Poisson process of rate $\alpha$.*

## 2.2 Taking care of lattice-type service times

As mentioned in the introduction, passing an arbitrary stationary and ergodic process $\mathbf{A}^1$ through a series of independent, identical ·/GI/∞ queues with lattice service times can result in non-Poisson process limits depending on whether the arrival process is also "lattice" with the same span or not.

If the arrival process $\mathbf{A}^1$ is "not lattice" with respect to the service process then again a Poisson limit occurs, as we shall presently show. However, the following example shows the difficulty involved in a direct implementation of the coupling scheme outlined in Proposition 1 to pair points of $\mathbf{A}^1$ with points of a Poisson process $\mathbf{P}^1$. Consider two "partners" $a$ and $p$ belonging to $\mathbf{A}^1$ and $\mathbf{P}^1$ respectively. Suppose their arrival times are $x$ and $x+3.6$. Suppose also that the service is lattice with span 1. As $a$ and $p$ progress through the series of queues, it is clear that although the random walk, $\tau^m$, associated with their inter-arrival times at the $m^{th}$ node may be recurrent, since the service is of lattice-type $|\tau^m|$ will *never ever* be less than 0.4. Hence the partners will never couple. Thus, we need to modify the coupling and pair only those customers that can come within $\delta$ of each other, where $\delta > 0$ is arbitrarily small. In order to proceed, we need the following definitions.

**Definition 2:** Given $c > 0$, $x$ $(mod\ c)$ is the element of $[0,\ c)$, which taken away from $x$, leaves an integer multiple of $c$.

Given a stationary point process $\mathbf{A} = \sum_{n=-\infty}^{\infty} \delta_{t_n}$, consider

$$\mathcal{P}^n(\omega) = \frac{1}{N^A([0,\ nc))} \sum_{t_i \in [0,\ nc)} \delta_{t_i\ (mod\ c)},$$

where $\delta_x$ is the point mass situated at $x$. For each $n$, $\mathcal{P}^n(\omega)$ can be considered as a probability measure on the compact space $[0,\ c]$ with the two end points, 0 and $c$, identified. To obtain $\mathcal{P}^n(\omega)$ we "wrap around" the points of $\mathbf{A}(\omega)$ in $[0,\ nc)$ about the interval $[0,\ c]$. The elementary lemma below shows that, for almost all $\omega$, $\mathcal{P}^n(\omega)$ converges (in the sense of weak convergence of distributions) to a random limit $\mathcal{P}^A(\omega)$ whenever $\mathbf{A}$ is a stationary point process of finite intensity.

**Lemma 1:** *Let $\mathbf{A} = \sum_{n=-\infty}^{\infty} \delta_{t_n}$ be a stationary point process of finite intensity and let*

$$\mathcal{P}^n(\omega) = \frac{1}{N^A([0,\ nc))} \sum_{t_i \in [0,\ nc)} \delta_{t_i\ (mod\ c)}.$$

*Then there exists a random probability measure $\mathcal{P}^A(\omega)$ on $[0,\ c]$ such that a.s. $\mathcal{P}^n(\omega) \stackrel{weakly}{\longrightarrow} \mathcal{P}^A(\omega)$.*

**Proof:** Let $f$ be a continuous function on $[0,\ c]$ with $f(0) = f(c)$. Define $F : \Omega \to \mathcal{R}$ by $F(\omega) = \sum_{t_i \in [0,\ c)} f(t_i)$. Then $F(\Theta_{jc}(\omega)) = \sum_{t_i \in [jc,\ (j+1)c)} f(t_i - jc)$. Hence

$$\int f\ d\mathcal{P}^n(\omega) = \frac{1}{N^A([0,\ nc))} \sum_{t_i \in [0,\ nc)} f(t_i\ (mod\ c)) = \frac{1}{N^A([0,\ nc))} \sum_{j=0}^{n-1} F(\Theta_{jc}(\omega))$$

Since A is $\Theta_t$-stationary, it is certainly stationary with respect to shifts $\{\Theta_{nc},\ n \in \mathcal{Z}\}$. It follows from the Ergodic Theorem and the finite intensity of A that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} F(\Theta_{jc}(\omega))$$

exists a.s. and is finite. Therefore $\lim_{n\to\infty} \int f_i\ d\mathcal{P}^n(\omega)$ exists a.s. and is finite for a countable dense set of functions $f_i$ on $[0,\ c]$ with $f_i(0) = f_i(c)$.

Since the sequence of probability measures $\mathcal{P}^n(\omega)$ is defined on the compact space $[0,\ c]$ with 0 and $c$ identified, they are tight. Hence there exist weak sub-sequential limits for the sequence $\mathcal{P}^n$. By the above argument, all the weak sub-sequential limits are identical. Let this limit be $\mathcal{P}^A$. This concludes the proof. ∎

**Discussion:** It should be noted that the limit in the above lemma can be random even when A is ergodic. This is because a function that is $\Theta_{nc}$-invariant need not be $\Theta_t$-invariant. For example, if A is the ergodic deterministic process of period 1 and $c = 1$, then $\mathcal{P}^A(\omega) = \delta_{X(\omega)}$, where $X(\omega)$ is uniformly distributed on $[0,\ 1]$. However, for ergodic A, $\mathcal{P}^A(\Theta_t(\omega)) = \Theta_{t\ (mod\ c)} \circ \mathcal{P}^A(\omega)$. To see this, simply note that

$$\begin{aligned}
&\mathcal{P}^A(\Theta_t(\omega)) \\
&= \lim_{n\to\infty} \frac{1}{N^A([t, t+nc))} \sum_{t_i \in [t, t+nc)} \delta_{t_i - t\ (mod\ c)} \\
&= \lim_{n\to\infty} \frac{N^A([0,nc))}{N^A([t,t+nc))} \frac{1}{N^A([0,nc))} \sum_{t_i \in [t,t+nc)} \Theta_{t\ (mod\ c)} \circ \delta_{t_i\ (mod\ c)} \\
&= \lim_{n\to\infty} \frac{1}{N^A([0,nc))} \Theta_{t\ (mod\ c)} \circ \sum_{t_i \in [kc,(k+n)c)} \delta_{t_i\ (mod\ c)}, \quad \text{where } t = kc + t\ (mod\ c), \\
&= \Theta_{t\ (mod\ c)} \circ \lim_{n\to\infty} \frac{1}{N^A([0,nc))} \sum_{t_i \in [kc,(k+n)c)} \delta_{t_i\ (mod\ c)} \\
&= \Theta_{t\ (mod\ c)} \circ \mathcal{P}^A(\omega).
\end{aligned}$$

Of course, since $\mathcal{P}^A(\Theta_t(\omega))$ is a measure on $[0,c]$ with the end points being identified, $\Theta_{t\ (mod\ c)} \circ \mathcal{P}^A(\omega)$ is understood to be a translate *mod* $c$ of $\mathcal{P}^A(\omega)$. By ergodicity of A, the orbit of $\{\Theta_t(\omega)\}$ as $t$ ranges over $\mathcal{R}$ is $\Omega$, making the limiting measure $\mathcal{P}^A$ almost surely equal to a uniform translation (*mod* $c$) of some measure $\nu$ on $[0,c]$.

**Definition 3:** A point process $A = \sum_{n=-\infty}^{\infty} \delta_{t_n}$ is said to be *uniform on* $[0,\ c]$ if

$$\frac{1}{N^A([0,\ nc))} \sum_{t_i \in [0,\ nc)} \delta_{t_i\ (mod\ c)}$$

converges almost surely to the uniform distribution on $[0,\ c]$ as $n \to \infty$.

We now obtain the following strengthening of Theorem 1.

**Theorem 2:** *Let* $\mathbf{A}^1$ *be a stationary and ergodic arrival process of rate* $\alpha$ *and let* $\mathbf{A}^n$ *be the departure process from the* $(n-1)^{th}$ *queue when* $\mathbf{A}^1$ *is fed through an infinite series of independent, identical* $\cdot/GI/\infty$ *queues. Then* $\mathbf{A}^n$ *converges in distribution to a Poisson process of rate* $\alpha$ *if and only if*
*(a) The service is non-lattice*
*or*
*(b) The process* $\mathbf{A}^1$ *is uniform on* $[0,\ c]$*, the span of the service times.*

**Proof:** Theorem 1 takes care of the Poisson convergence when (a) is satisfied.

To prove (b), we use the same argument as in Theorem 1, except that we must change the rules for matching customers. This time given a continuous function $f$ with compact support (support contained in $[0, N]$, say), $\epsilon$ (a measure of how close the matched customers should be in order to be declared "coupled"), and $c$ the span of the service time, we divide time into large blocks of length $Rc$ ($R$ is an integer). We then divide $(0,\ c]$ into $1/\epsilon$ intervals of length $\epsilon c$ (without loss of generality $1/\epsilon$ is an integer), and choose $R$ so large that outside a set of probability $1-\epsilon$, the number of points of $\mathbf{A}^1$ and $\mathbf{P}^1$ in $(0,\ Rc]$ which are in

$$\bigcup_{k=0}^{R-1} [kc + i\epsilon c,\ kc + (i+1)\epsilon c) \quad \text{for} \quad i = 0, 1, \cdots, (1/\epsilon - 1)$$

are both between $Rc(\alpha\epsilon - \frac{\epsilon}{2})$ and $Rc(\alpha\epsilon + \frac{\epsilon}{2})$. That is, we take the realizations of $\mathbf{A}^1$ and $\mathbf{P}^1$ over the interval $(0,\ Rc]$ and "wrap" them around $(0,\ c]$. If $R$ is big enough, because $\mathbf{A}^1$ and $\mathbf{P}^1$ are both uniform over $c$, the distribution of points in $(0,\ c]$ after wrapping around will be close to uniform with a high probability and on the average there will be $Rc\alpha\epsilon$ points in the above union of intervals. Hence a suitably large choice of $R$ gives the desired minimum of $Rc(\alpha\epsilon - \frac{\epsilon}{2})$ points with probability greater than $1-\epsilon$.

As before we choose an independent uniform $U$ on $[0,\ Rc]$ and we call an interval $(jRc + U,\ (j+1)Rc + U]$ "good" if the number of points of $\mathbf{A}^1$ and $\mathbf{P}^1$ in $(jRc + U,\ (j+1)Rc + U]$ which are in

$$\bigcup_{k=0}^{R-1} [jRc + U + kc + i\epsilon c,\ jRc + U + kc + (i+1)\epsilon c) \quad \text{for} \quad i = 0, 1, \cdots, (1/\epsilon - 1)$$

are both within $Rc(\alpha\epsilon - \frac{\epsilon}{2})$ and $Rc(\alpha\epsilon + \frac{\epsilon}{2})$. We match up points in $\mathbf{A}^1$ and $\mathbf{P}^1$ as follows:

1) If the interval $(jRc + U,\ (j+1)Rc + U]$ is not good, then no customers of $\mathbf{A}^1$ and $\mathbf{P}^1$ are matched.

2) If the interval $(jRc + U,\ (j+1)Rc + U]$ is good, and the arrival times of the customers of $\mathbf{A}^1$ and $\mathbf{P}^1$ in

$$\bigcup_{k=0}^{R-1} [jRc + U + kc + i\epsilon c,\ jRc + U + kc + (i+1)\epsilon c)$$

for each $i = 0, 1, \cdots, (1/\epsilon - 1)$ are, respectively, $x_1^{ij} < x_2^{ij} < \cdots < x_{\lfloor Rc(\alpha\epsilon - \frac{\epsilon}{2})\rfloor}^{ij} < \cdots,$ and

$y_1^{ij} < y_2^{ij} < \cdots < y^{ij}_{\lfloor Rc(\alpha\epsilon - \frac{\epsilon}{2})\rfloor} < \cdots$, then we match up the customer at $x_l^{ij}$ with the customer at $y_l^{ij}$ for $l \leq \lfloor Rc(\alpha\epsilon - \frac{\epsilon}{2})\rfloor$.

The rest of the proof is the same as in part (a) with the recurrent and transient cases being dealt with appropriately.

It remains to show that if both conditions (a) and (b) fail then $\mathrm{A}^n$ cannot tend to a Poisson process of rate $\alpha$ in distribution. However, should (a) fail, then the service time has a distribution with span $c$, say. If (b) fails, then $\mathrm{A}^1$ is non-uniform on $[0,\ c]$. But Lemma 1 tells us that $\mathcal{P}^A(\omega) = \lim_{n\to\infty} \frac{1}{N^A([0,\ nc))} \sum_{t_i \in [0,\ nc)} \delta_{t_i\ (mod\ c)}$ exists a.s. As noted in the discussion following Lemma 1, $\mathcal{P}^A(\Theta_t\omega) = \Theta_{t\ (mod\ c)} \circ \mathcal{P}^A(\omega)$. Therefore the event that $\mathcal{P}^A$ is uniform is translation invariant. As $\mathrm{A}^1$ is ergodic, this event must have probability 0 or 1. By similar reasoning if $\mathcal{P}^A$ is not a.s. uniform then it must be a.s. equal to some distinct distribution $\nu$ on $[0,\ c]$ or a translation ($mod\ c$) of $\nu$. But in this case Theorem 3 below shows that $\mathrm{A}^n$ cannot tend to a Poisson point process in distribution. ■

Given a probability measure $\nu$ on $\mathcal{R}$ with support in $[0,\ c)$, define the measures $\nu^n, n \in \mathcal{Z}$ with support in $[nc,\ (n+1)c)$ as translates of $\nu$ by $nc$ units of time. That is $\nu^n(S) = \nu(S\ (mod\ nc))$ for Borel sets $S$ in $[nc,\ (n+1)c)$. Note that $\nu^0 = \nu$.

**Definition 4:** Given $c > 0$ and a measure $\nu$ on $[0,\ c)$, the process $\mathrm{P}_\nu$ is said to be a *$\nu$-Poisson process of rate $\alpha$* if the following conditions hold.
i) The number of points of $\mathrm{P}_\nu$ in $[nc,\ (n+1)c)$ is an i.i.d. sequence, as $n$ varies over $\mathcal{Z}$, with each marginal being distributed as a Poisson, parameter $\alpha c$, random variable.
ii) Each point of $\mathrm{P}_\nu$ in $[nc,\ (n+1)c)$ is distributed over the interval $[nc,\ (n+1)c)$ according to $\nu^n$, independent of all other points.

Note that according to the above definition $\mathrm{P}_\nu$ is the usual Poisson process with parameter $\alpha$ if $\nu$ is the uniform distribution over $[0,\ c)$, and it is a batch process of i.i.d. Poisson random variables if $\nu$ is a point mass at $x \in [0,\ c)$. Hence $\mathrm{P}_\nu$ is, in general, not a time stationary process. However, by shifting the paths of $\mathrm{P}_\nu$ uniformly over the interval [0, c), we obtain a *stationary $\nu$-Poisson process.*

**Theorem 3:** *Let $\mathrm{A}^1 = \sum_{n=-\infty}^{\infty} \delta_{t_n}$ be a stationary and ergodic process of rate $\alpha$ and suppose*

$$\lim_{n\to\infty} \frac{1}{N^{A^1}([0,\ nc))} \sum_{t_i \in [0,\ nc)} \delta_{t_i\ (mod\ c)}$$

*converges a.s. to a translate ($mod$ c) of some distribution $\nu$ on [0, c). Let $\mathrm{A}^n$ be the departure process from the $(n-1)^{th}$ stage of an i.i.d. series of $\cdot$/GI/$\infty$ queues when $\mathrm{A}^1$ is inputted to the first queue. Suppose the service time of each customer is lattice with span c. Then $\mathrm{A}^n$ converges in distribution to a stationary $\nu$-Poisson process of rate $\alpha$.*

**Proof:** Again we couple $\mathrm{A}^1$ with a stationary $\nu$-Poisson process $\mathrm{P}^1$. (In Lemma 2 we show that $\mathrm{P}^1$ is an invariant distribution for a lattice-type $\cdot$/GI/$\infty$ queue.) First we note that the ergodicity of $\mathrm{A}^1$ implies that there exists a unique probability distribution $\nu$ on $[0, c)$ such that a.s.

$$\lim_{n\to\infty} \frac{1}{N^{A^1}([0,\ nc))} \sum_{t_i \in [0,\ nc)} \delta_{t_i\ (mod\ c)}$$

converges to a translate ($mod$ c) of $\nu$. We construct our stationary $\nu$-Poisson process $\mathrm{P}^1$ from

$A^1$ (after possibly enlarging our probability space). This is different form previous couplings where the two point processes $A^1$ and $P^1$ were chosen to be independent. Given $A^1$ let $\nu(\omega)$ be the limit of the random measures

$$\lim_{n\to\infty} \frac{1}{N^{A^1}([0,\ nc))} \sum_{t_i \in [0,\ nc)} \delta_{t_i\ (mod\ c)}$$

and let $P^1$ be a $\nu$-Poisson process, conditionally independent of $A^1$ given $\nu$. Then it is clear that $P^1$ is a stationary $\nu$-Poisson process. Fix $\epsilon$ arbitrarily small but positive. We pick integer $R$ to be sufficiently large that outside of probability $1-\epsilon$, the number of points of $A^1$ and $P^1$ in $[0,\ Rc)$ which are in

$$\bigcup_{k=0}^{R-1} [kc+i\epsilon c,\ kc+(i+1)\epsilon c) \quad \text{for} \quad i=0,1,\cdots,(1/\epsilon-1)$$

are both between $Rc\alpha\left(\nu(\omega)([i\epsilon,\ (i+1)\epsilon)) - \frac{\epsilon}{2}\right)$ and $Rc\alpha\left(\nu(\omega)([i\epsilon,\ (i+1)\epsilon)) + \frac{\epsilon}{2}\right)$. This $R$ can of course be chosen in a non-random way. The proof now proceeds exactly as in part (b) of Theorem 2. ∎

**Lemma 2:** *Consider a $\cdot/GI/\infty$ queue whose service is lattice with span 1. If $\nu$ is a probability measure on $[0,\ 1]$, then a $\nu$-Poisson process of rate $\alpha$, $P_\nu$, is an invariant distribution for this queue.*

**Proof:** The proof is based on the fact that any two $\cdot/GI/\infty$ queues are interchangeable. (Two queueing nodes in tandem are said to be *interchangeable* if, for arbitrary arrival processes, the law of the overall departure process from the tandem is invariant with respect to the ordering of the two nodes. It is obvious that any two $\cdot/GI/\infty$ queues are interchangeable.) Let $G_1$ and $G_2$ be two $\cdot/GI/\infty$ queues and let $\sigma^i$ denote the service time of a typical customer at node $i$, $i=1,2$. Suppose the laws of $\sigma^i$ are given by

$$P(\sigma^1 = k) = p_k,\ k=0,1,\cdots; \qquad P(\sigma^2 \in S) = \nu(S),$$

where $S$ is a Borel subset of $[0,\ 1]$. We are required to show that $G_1(A) = A$.

We first show that a batch arrival process B, of rate $\alpha$, composed of i.i.d. Poisson random variables living on the integers is invariant to the queue $G_1$. To see this, suppose that B is represented by the i.i.d. sequence of Poisson random variables $\{X_n,\ n\in \mathcal{Z}\}$ with $E(X_n)=\alpha$. Split each $X_n$ into random variables $\{X_n^k\}_{k\geq 0}$ by sampling $X_n$ in an i.i.d. fashion according to the probabilities $p_k$. The interpretation is that $X_n^k$ is the number of customers arriving at time $n$ and requiring $k$ units of service. Because $X_n$ is Poisson distributed each $X_n^k$ is a Poisson random variable with parameter $\alpha p_k$; and, for distinct $k$ and $j$ $X_n^k$ and $X_n^j$ are independent. Therefore the members of the doubly indexed family of random variables $\{X_n^k,\ n\in\mathcal{Z}, k\geq 0\}$ are mutually independent. The departure process from the node $D = \{Y_n\}$ is given by $Y_n = \sum_{k=0}^{\infty} X_{n-k}^k$. The claim now follows since each $Y_n$ is a Poisson random variable of rate $\alpha$ (being the sum of independent Poisson random variables of rate $\alpha p_k$), and for distinct $k$ and $j$, $Y_k$ and $Y_j$ are independent. Therefore, $G_1(B) = B$.

Notice that by definition, $A = G_2(B)$. By interchangeability,

$$G_1(G_2(B)) = G_2(G_1(B)).$$

But the left-hand side equals $G_1(A)$ and the right-hand side equals $G_2(B) = A$. Hence $G_1(A) = A$ and the lemma is proved. ∎

**Corollary:** *Given $\nu$, a probability measure on* $[0,\ 1]$, *a stationary $\nu$-Poisson process is an invariant distribution for a $\cdot/GI/\infty$ queue whose service is lattice with span 1.*

**Remark** The method we have advanced can also be easily extended to the case when the queues in series are not i.i.d. For instance, if the service times have the property that the service time of every customer converges almost surely to zero, then the series of queues has the property that *all* stationary ergodic inputs converge to a Poisson if and only if the sums of symmetrized service times, $\sum_{k=1}^{N}(\sigma_k^a - \sigma_k^p)$, do not converge. This follows from the celebrated 3-series Theorem of Kolmogorov. To see how this applies to our case, note that if the symmetrized service sums do not converge, then by elementary martingale arguments it follows that the sums must have liminf equal to minus infinity and limsup equal to infinity. This, together with the assumed property that service times tend to zero almost surely implies that the coupling of matched customers must be successful.

Conversely if the symmetrized sums converge, then for each customer, $x$, there is a random variable $X(x) < \infty$ a.s. and a deterministic sequence of numbers, $c(n)$, not depending on $x$, so that for all $n$ |service time of $x$ at the first $n$ queues $- c(n)| < X(x)$. Thus if the initial arrival process is a sufficiently extreme mixture of long vacant periods alternating with short dense batches of arrivals, then no Poisson convergence is possible.

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